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Parametric General Solutions of Boolean Equations Via Variable–Entered Karnaugh Maps

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Abstract. A new method for obtaining a compact parametric general solution of a system of Boolean equations is presented. The method relies on the use of the variable-entered Karnaugh map (VEKM) to implement various steps of the solution procedure and to ensure minimization of the expressions obtained. It is highly efficient as it requires the construction of *natural* maps that are significantly *smaller* than those required by classical methods. Moreover, the method is applicable to general Boolean equations and is not restricted to the two-valued case. As an offshoot, the paper contributes some pictorial insight on the representation of "big" Boolean algebras and functions. It also predicts the correct number of particular solutions of a Boolean equation, and produces a comprehensive list of particular solutions, if desired. Details of the method are carefully explained and further demonstrated via an illustrative example.
Key Words. Boolean equations, Parametric general solutions, Particular solutions, Variable-entered Karnaugh maps.

1. Introduction

The topic of Boolean equations has been a hot topic of research for almost two centuries and its importance can hardly be overestimated. Boolean-equation solving permeates many areas of modern science such as logical design, biology, grammars, chemistry, law, medicine, spectroscopy, and graph theory [1, 2]. Many important problems in operations research can be reduced to the problem of solving a system of Boolean equations. The solutions of Boolean equations serve also as an important tool in the treatment of pseudo-Boolean equations and inequalities, and their associated problems in integer linear programming [3].

To solve a system of n Boolean equations, the equations are usually combined into an equivalent single Boolean equation whose set of solutions is exactly the same as that of the original system of equations. This is conceptually simpler than obtaining the set of solutions for each equation and then forming the intersection of such sets to obtain the set of solutions of the overall system. Typically, general subsumptive or parametric solutions are sought [1-5]. Such general solutions are compact forms from which an exhaustive enumeration of particular solutions can be readily obtained. All types of solutions are obtainable by either algebraic or map methods [1, 2, 6-10].

Both Brown [1, 2] and Tucker and Tapia [6, 7] solved Boolean equations using a Marquand diagram (also called Veitch chart) in which natural binary order is followed, or used a Conventional Karnaugh Map (CKM) in which the rows and columns are arranged according to a reflected binary code. Their methods in [1, 2, 6, 7] are restricted to (or reduce their subject to) two–valued Boolean variables. All these methods state their rules in cell-wise tabular form.

Rushdi [8, 10] developed yet another mapping method for obtaining a subsumptive general solution of a system of Boolean equations. This method is not restricted to the two-valued case and requires the construction of maps that are significantly smaller than those required by earlier procedures. This is because it relies on the use of a more powerful map, namely the variable-entered Karnaugh map (VEKM). The VEKM is an adaptation of the CKM that retains most of its pictorial insight and effectively combines algebraic and mapping techniques. Historically, the VEKM was developed to double the variable-handling capability of the CKM [11]. Later, the VEKM was shown to be the direct or natural map for finite Boolean algebras other than the bivalent or 2-valued Boolean algebra (switching algebra) [1, 8, 10, 12, 13]. These algebras are sometimes called ‘big’ Boolean algebras, and are useful and unavoidable, even if unrecognizable, in many applications [1].

In the present work, we develop a powerful VEKM method to implement an existing procedure [1, 2] for deriving the parametric general solution of a system of Boolean equations. This VEKM method is more efficient than the CKM method in [2], since it requires significantly smaller maps. Another merit of this VEKM is that it produces the solution in the most compact form, thanks to a well-known VEKM minimization procedure [11-13]. As an offshoot, the present work contributes some pictorial insight on the representation of “big” Boolean algebras and functions. Moreover, it correctly predicts the number of particular solutions of the pertinent system of equations, and it identifies a pitfall in an earlier attempt at such a prediction in [2]. If desired, the compact parametric solution obtained is expanded into an exhaustive list of particular solutions.

The rest of this paper is organized as follows. Section II introduces some notation, while Section III reviews pertinent properties of Boolean algebras and addresses the question of pictorial representations of Boolean algebras and functions. Section IV presents a new VEKM method for deriving the parametric solution of a Boolean equation, which is a map adaptation of an existing algebraic procedure. Section V demonstrates the VEKM method via a big-Boolean-algebra example. Section VI addresses the topic of the number of particular solutions and their exhaustive listing. Section VII concludes the paper.

2. Representations of Boolean Algebras and Functions

A Boolean algebra is a quintuple $B = (\mathbf{B}, \vee, \wedge, 0, 1)$ in which **B** is a set, called the carrier; $\vee$ and $\wedge$ are binary operations on **B** and the zero (0) and unit (1) elements are distinct members of **B**, with certain postulates on commutativity, distributivity, binary-operation identities and complementation being satisfied [1-5]. The following facts about a Boolean algebra can be deduced :

1. Every element X of **B** has a unique complement $\bar{X}$.
2. There is a partial-order or inclusion ($\leq$) relation on B that is reflexive, anti-symmetric, and transitive.

3. A Boolean algebra *B* enjoys many useful properties such as associativity, idempotency, absorption, involution, consensus and duality.

4. A Boolean algebra *B* is a complemented distributive lattice with distinct 0 and 1 elements.

5. A nonzero element *Z* of **B** is said to be an atom of **B** iff for every $X \in \mathbf{B}$, the condition $Z \leq X$ implies that $X = Z$ or $X = 0$.

6. Every finite Boolean algebra *B* is atomic, i.e. for every nonzero element $X \in \mathbf{B}$, there is some atom *Z* such that $Z \leq X$.

7. Examples of Boolean algebras include the algebra of classes (subsets of a set), the algebra of propositional functions, the arithmetic Boolean algebra (where $\vee$ and $\wedge$ denote "the least common multiple" and "the greatest common divisor"), as well as the switching or two-element Boolean algebra.

8. Boolean algebras with the same number of elements are isomorphic.

9. Every finite Boolean algebra *B* has 2^m element, where m is the cardinality of (number of elements in) the set of atoms of **B**.

An n-variable function $f: \mathbf{B}^n \rightarrow \mathbf{B}$ is Boolean iff it can be expressed in the minterm canonical form

$$f(\mathbf{X}) = \bigvee f(\mathbf{A})\ \mathbf{X}^{\mathbf{A}}, \tag{1}$$

where the ORing operation $\vee$ in (1) extends over all $\mathbf{A} = (a1, a2, \ldots, an) \in \{0, 1\}^n$, and $\mathbf{X}^{\mathbf{A}}$ is defined for $\mathbf{X} = (X_1, X_2, \ldots, X_n) \in \mathbf{B}^n$ as

$$\mathbf{X}^{\mathbf{A}} = X_1^{a1} X_2^{a2} \ldots\ldots X_n^{an}, \tag{2}$$

where for $X \in \mathbf{B}$ and $a \in \{0,1\}$, X^a is defined by

$$X^0 = \bar{X}, \tag{3a}$$

$$X^1 = X. \tag{3b}$$

In (1), the function values $f(\mathbf{A})$, where $\mathbf{A} \in \{0, 1\}^n$ are called the discriminates of $f(\mathbf{X})$.

A dramatic consequence of (1) is that a Boolean function $f: \mathbf{B}^n \rightarrow \mathbf{B}$, where **B** is a carrier of 2^m elements, is uniquely determined by a truth table or map partially representing *f* for the restricted domain $\{0, 1\}^n$ which is a strict subset of the complete domain B^n. Such a representation needs 2^n table or map cells. The complete function table of *f* (which covers the complete domain B^n, and hence requires $(2^m)^n = 2^{mn}$ table lines or map cells) is neither warranted nor needed. In fact, such a complete table should be avoided. Not only does it require tedious work, but it also poses a problem of checking consistency in its $B^n/\{0, 1\}^n$ part.

It is customary to name the elements of **B** in terms of a minimum number of abstract variables $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_k)$. Usually the elements of **B** are taken as the elements of the free Boolean algebra $FB(\mathbf{Y}) = FB(Y_1, Y_2, \ldots, Y_k)$. $FB(\mathbf{Y})$ is isomorphic to the Boolean algebra of switching functions of k variables, and has 2^{2^k} elements.

This method works directly when $m = 2^k$. It is also applicable when $2^{k-1} < m < 2^k$, provided that certain constraints are added to nullify some atoms of $FB(\mathbf{Y})$, i.e., some mini-terms of the switching algebra isomorphic to $FB(\mathbf{Y})$.

The first two Boolean carriers are $\mathbf{B}_2 = \{0, 1\}$ and $\mathbf{B}_4 = \{0, 1, a, \bar{a}\}$. Let us temporarily jump to the fourth Boolean carrier $\mathbf{B}_{16} = FB(a, b)$ whose elements constitute the hypercube lattice shown in Fig. (1). This carrier has 4 atoms ab, $\bar{a}b$, $a\bar{b}$ and $\bar{a}\bar{b}$ which constitute the 4 dimensions of the hypercube. Now, if one of these atoms is nullified (e.g., if we set $a\bar{b}$ to 0), the hypercube in Fig. (1) loses one of its four dimensions, and collapses into the cube of Fig. (2), which constitutes the third Boolean carrier $\mathbf{B}_8$. Similarly, $FB(Y_1, Y_2, Y_3)$ can be used to represent $\mathbf{B}_{256}$ directly, or to represent $\mathbf{B}_{32}$, $\mathbf{B}_{64}$, $\mathbf{B}_{128}$ with 3, 2, and 1 atomic constraints respectively.

We now consider the question: What is the natural map for $f(\mathbf{X}): \mathbf{B}^n \rightarrow \mathbf{B}$, with **B** having 2^m elements where $2^{k-1} < m \leq 2^k$. The restricted input domain for this function is $\{0, 1\}^n$ representing only the bivalent values of $\mathbf{X} = (X_1, X_2, \ldots, X_n)$. The entries of the map could by any of the 2^{2^k} switching functions of $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_k)$, with possible ORing with don't care terms representing the nullified atoms when $m < 2^k$. The map described this way is nothing but the Variable-Entered Karnaugh Map (VEKM) typically used to represent an incompletely-

specified switching function $f(\boldsymbol{X};\boldsymbol{Y})$: $\{0, 1\}^{n+k} \rightarrow \{0, 1\}$ using n map variables **X** and k entered variables **Y** [8, 10-13].

Note that instead of a conventional map representation of the switching function of (n+k) variables (X; Y) as f (X; Y) $\{0, 1\}^{n+k} \rightarrow \{0, 1\}$ which has constant {0, 1} entries , a VEKM divides the input variables into n map variables **X** and k entered ones **Y**, and therefore represents the function $f(\boldsymbol{X})$: $\{0, 1\}^{n} \rightarrow \{0, 1\}^{k}$ with variable entries belonging to $FB(\mathbf{Y})$. Therefore, a VEKM is the "natural" map for representing $f(\boldsymbol{X})$: $\mathbf{B}^{n} \rightarrow \mathbf{B}$ where **B** has 2^{m} elements and $2^{k-1} < m \leq 2^{k}$. In other words, the VEKM is the natural map for a Boolean function $f(\boldsymbol{X})$, since its input combinations is the restricted domain of f and its entries cover the range of f. An attempt to represent a big Boolean function via a conventional Karnaugh maps (CKM) should be discouraged. Such a representation is a) less efficient, as it produces significantly larger maps, and b) conceptually misleading, as it inadvertently shifts symbols used in describing the output of a function into extraneous inputs, and at the same time conceals the actual nature of this output by reducing it into a bivalent form.

A conventional Karnaugh map is conceptually meaningful however, for solving a switching equation involving the function $f(\boldsymbol{X}; \boldsymbol{Y})$: $B_2^{n+k} \rightarrow B_2$, when it is required to express a set of dependent switching variables **X** in terms of a set of independent switching variables **Y** [14].

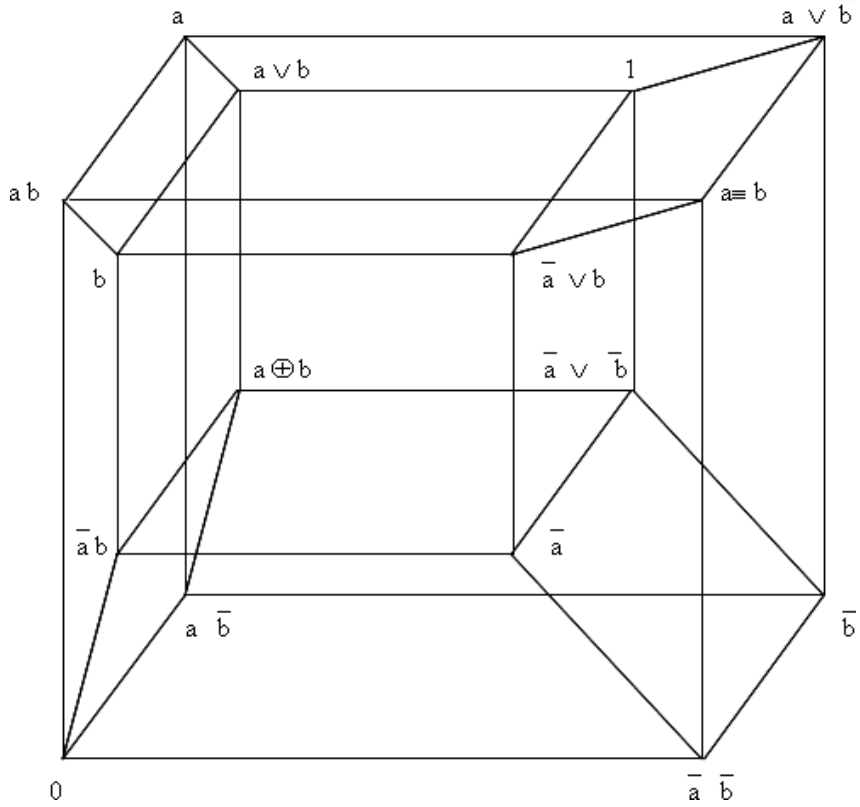


Fig. (1). A lattice indicating the partial ordering among the 16 elements of B_{16}.

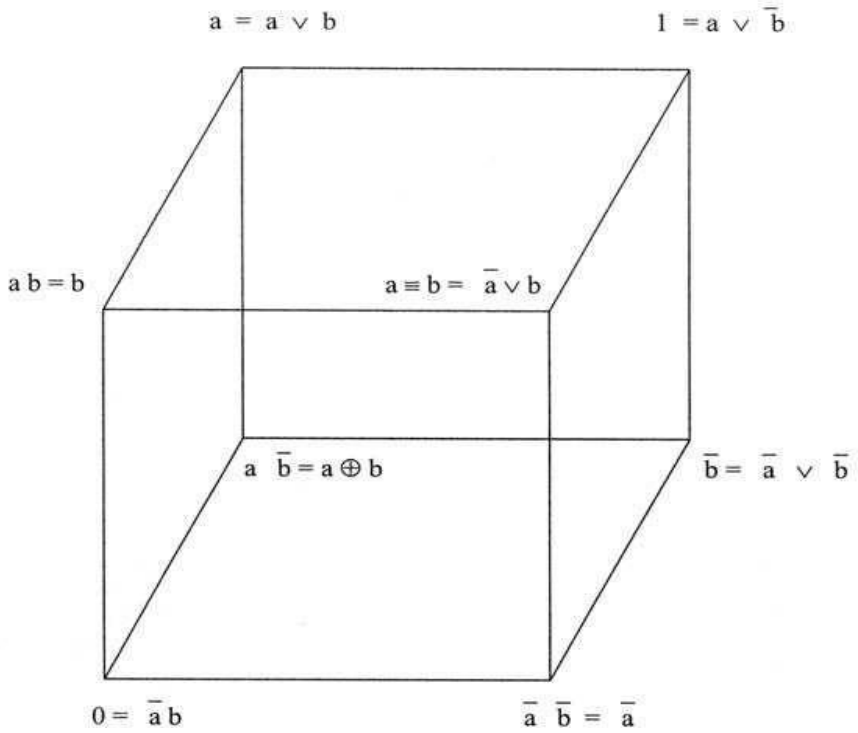


Fig. (2). The lattice in Fig. (1) when collapsed under the condition $\bar{a}\,b = 0$.

3. VEKM Derivation of Parametric Solutions

Brown [1, 2] proved that n parameters are sufficient to construct a parametric general solution of an n-variable Boolean equation $g(\mathbf{X}) = 1$, where g: $\mathbf{B}^n \rightarrow \mathbf{B}$. He proposed a procedure for constructing such a solution using the fewest possible parameters, $p_1, p_2, \ldots, p_k$, which are elements of **B**, where $k \leq n$. We adapt this procedure of Brown into a VEKM procedure as follows:

1. Construct a VEKM representing $g(\mathbf{X})$. Such a construction is achieved via a Boole-Shannon tree expansion [11-13]. If the original Boolean equation is in the dual form $f(\mathbf{X}) = 0$, then construct a VEKM for $f(\mathbf{X})$, and complement it cell-wise [15] to obtain a VEKM for $\bar{f}(X) = g(X)$.

2. Expand the entries of the VEKM of $g(\mathbf{X})$ as ORing of appropriate atoms of the Boolean carrier **B,** or equivalently as a minterm expansion of the free Boolean algebra representing **B.**

3. If certain atoms of **B** do not appear at all in any cell of the VEKM for $g(\mathbf{X})$, then these atoms must be forbidden or nullified. Such nullification constitutes *a consistency condition* for the given Boolean equation.

4. Construct a VEKM for an associated function $G(X_1, X_2, \ldots, X_n; p_1, p_2, \ldots, p_k)$. This VEKM is deduced from that of $g(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n)$ through the following modifications:

a) Each appearance of an entered atom in the VEKM of g is ANDed with a certain element of a set of orthonormal tags of minimal size. Table (1) gives examples of such sets as a function of the number of cells in which an atom appears. An orthonormal set consists of a set of terms T_i, i = 1, 2, …, k, which are both exhaustive ($T_1 \vee T_2 \vee \ldots \vee T_k = 1$) and mutually exclusive ($T_i T_j = 0$ for $1 \leq i < j \leq k$).

b) Each nullified atom is entered don't care in all the VEKM cells.

5. The parametric solution is

X_i = The sum (ORing) of the 2^{n-1} cells constituting that half of the VEKM in which $\mathbf{X}_i$ is asserted

$$(\mathbf{X}_i = 1), \quad i = 1, 2, \ldots, n. \tag{4}$$

6. Apply an appropriate VEKM minimization procedure [11-13] to recast (4) in a minimal form.

Table (1). Minimal orthonormal sets of tags attached to instances of each atom depending on the number of cells in which it appears (Each orthonormal set consists of exhaustive and mutually exclusive terms).

The No. of cells in which an atom appears	A set of orthonormal tags of minimal size	The minimum number of parameters required
1	$\{1\}$	0
2	$\{\bar{p}_1, p_1\}$	1
3	$\{\bar{p_1}, p_1\bar{p_2}, p_1p_2\}$	2
4	$\{\bar{p}_1\bar{p}_2, \bar{p}_1p_2, p_1\bar{p}_2, p_1p_2\}$	
5	$\{\bar{p_1}\bar{p_2}, \bar{p_1}p_2, p_1\bar{p_2}, p_1p_2\bar{p}_3, p_1p_2p_3\}$	
6	$\{\bar{p_1}\bar{p_2}, \bar{p_1}p_2, p_1\bar{p}_2\bar{p}_3, p_1\bar{p}_2p_3, p_1p_2\bar{p}_3, p_1p_2p_3\}$	3
7	$\{\bar{p_1}\bar{p_2}, \bar{p_1}p_2\bar{p_3}, \bar{p_1}p_2p_3, p_1\bar{p_2}\bar{p_3}, p_1\bar{p_2}\bar{p_3}, p_1p_2\bar{p_3}, p_1p_2p_3\}$	
8	$\{\bar{p}_1\bar{p}_2\bar{p}_3, \bar{p}_1\bar{p}_2p_3, \bar{p}_1p_2\bar{p}_3, \bar{p}_1p_2p_3, p_1\bar{p}_2\bar{p}_3, p_1\bar{p}_2\bar{p}_3, p_1p_2\bar{p}_3, p_1p_2p_3\}$	

4. A Big Boolean-algebra Illustrative Example

Let us apply the present method in terms of a VEKM representation to find the parametric general solution of an equation of the form $g(X_1, X_2, X_3) = 1$, where g is a Boolean function $g: B_{16}^3 \rightarrow \mathbf{B}_{16}$ and $\mathbf{B}_{16}$ is the Boolean carrier of 16 elements shown in Fig. (1). Note that the complete input space of g consists of $16^3 = 4096$ combinations of **X**, but g is uniquely defined by the values assigned to it on only 8 combinations of **X**, namely those belonging to $\{0, 1\}^3$. Let g be given by the formula

$$g(\mathbf{X}) = bX_1\bar{X}_2X_3 \vee ab\bar{X}_1X_2 \vee ab\bar{X}_1\bar{X}_2 \vee abX_1X_2 \vee a\bar{X}_1\bar{X}_2\bar{X}_3 = 1. \quad (5)$$

Equation (5) has been solved by Brown [2]. His first step is to expand g not only with respect to the 3 variables X_1, X_2, and X_3 but further with respect to the "constants" a and b thereby producing a 5-variable 32-cell Karnaugh map. In our present procedure, however, we expand g only with respect to the true variables X_1, X_2, and X_3, thereby representing g by the 8-cell VEKM in Fig. (3) which has X_1, X_2, and X_3 as map variables and has a and b as entered "variables". Since a and b are actually constants, this VEKM is a natural map for g. Now, Fig. (4) shows each of the entries of the map of g in Fig. (3) being expanded as ORing of appropriate atoms of $\mathbf{B}_{16}$, or equivalently as a sum of certain minterms of FB(a, b). Out of the 4 atoms of $\mathbf{B}_{16}$, three atoms, namely, ab, $a\bar{b}$, and $\bar{a}b$ appear somewhere in the cells of the map of Fig. (4), while the fourth atom ($\bar{a}\bar{b}$) does not appear at all therein. This means that the atom $\bar{a}\bar{b}$ is nullified or forbidden ($\bar{a}\bar{b} = 0$). In Fig. (5), the function $g(X_1, X_2, X_3)$ is replaced by an associated function $G(X_1, X_2, X_3, p_1, p_2)$, where each appearance of an entered atom in Fig. (4) is ANDed with a certain element of a set of orthonormal tags of minimal size (See Table 1). The atom $a\bar{b}$ appears in 4 of the 8-cells of the map in Fig. (4), so each of these 4 appearances is tagged with a particular element of the orthonormal set $\{\bar{p_1}\bar{p_2}, \bar{p_1}p_2, p_1\bar{p_2}, p_1p_2\}$, respectively. The atom $\bar{a}b$ appears in 3-cells of the map of Fig. (4), and hence each of these 3 appearances is ANDed with its own tag selected from the orthonormal set $\{\bar{p}_1, p_1\bar{p}_2, p_1p_2\}$. Finally, the atom ab made its appearance 3 times in the cells of the map of Fig. (4), and utilized a set of orthonormal tags equivalent to that of the atom $\bar{a}b$. In assigning tags to various instances of various atoms, slight simplifications are achieved if adjacent tags are located in adjacent cells whenever possible. The map in Fig. (5) is now completed by adding the nullified atom ($\bar{a}\bar{b}$) as a don't-care entry in each of the 8-cells of the map.

Now, the parametric solution can be readily deduced from Fig. (5). For i = 1, 2, and 3, the variable X_i is equal to the sum (ORing) of the 4-cells constituting half of the map for which X_i is asserted, i.e., for which $X_i = 1$. This means that

$$X_1 = a\bar{b}\,p_1p_2 \vee a\bar{b}\,p_1\bar{p}_2 \vee ab\;p_1p_2 \vee ab\;p_1\bar{p}_2 \vee d\,(\bar{a}b)\,, \tag{6a}$$

$$X_2 = ab\;p_1p_2 \vee ab\;p_1\bar{p}_2 \vee ab\;p_1p_2 \vee ab\;p_1\bar{p}_2 \vee\; d\,(\bar{a}b)\,, \tag{6b}$$

$$X_3 = a\bar{b}\;p_1p_2 \vee ab\;p_1p_2 \vee ab\;p_1p_2 \vee ab\;p_1 \vee ab\;p_1p_2 \vee d(\bar{a}b)\,, \tag{6c}$$

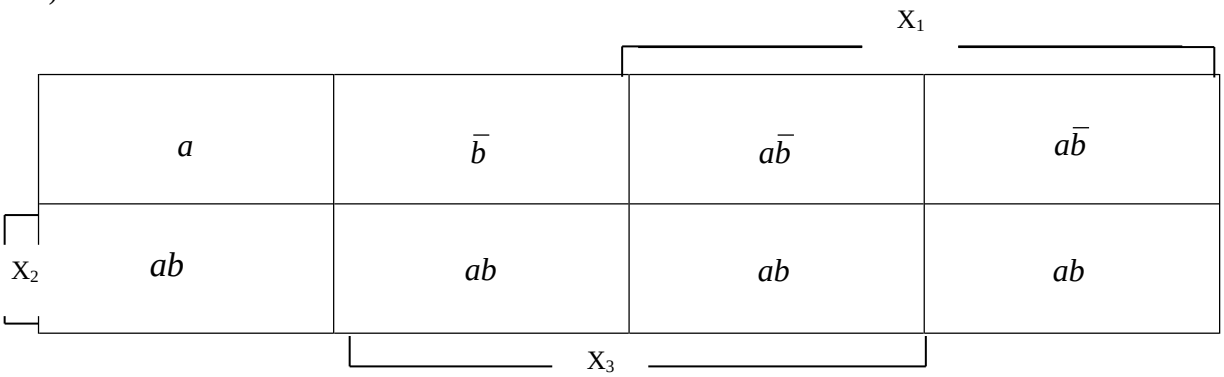
The expressions (6) can be minimized by any algebraic or map method, such as the VEKM minimization procedure in [11-13]. Due to space limitations, we do not elaborate on this procedure herein, though we give the reader a glimpse of its details in Fig. (6) and Fig. (7). Each of these two figures shows pictorially and succinctly how the VEKM minimization procedure achieves minimal expressions for X_1, X_2, and X_3. These minimal expressions are

$$X_1 = p_1\bar{b}, \tag{7a}$$

$$X_2 = p_1(\bar{a} \vee b)\,, \tag{7b}$$

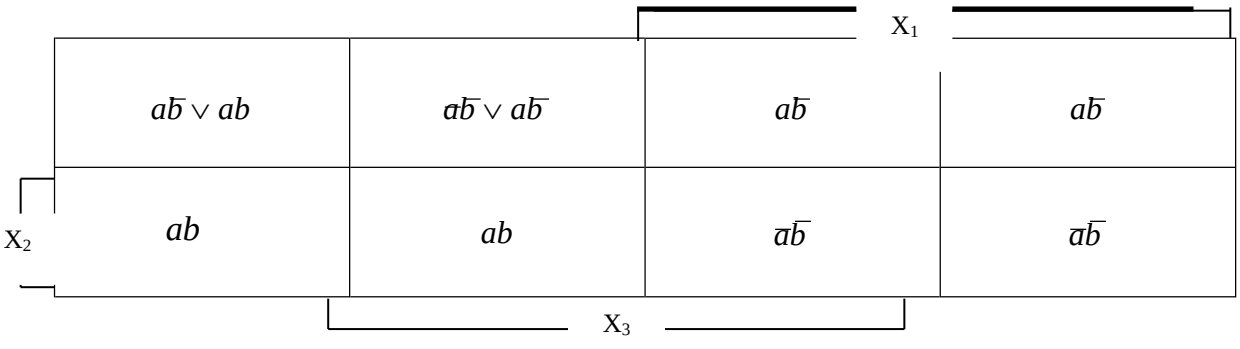
$$X_3 = p_2\bar{b} \vee p_1p_2 \vee \bar{p}_1\bar{a}\,, \tag{7c}$$

In the general solution (7), the two parameters p_1 and p_2 are independently chosen elements of the underlying Boolean algebra, i. e., of the $\mathbf{B}_{16}$ carrier in Fig. (1) that collapsed into a $\mathbf{B}_8$ carrier in Fig. (2). Since each of p_1 and p_2 can assume one out of eight possible values, the total number of combinations of values for p_1 and p_2 is 8^2 or 64 values. This number is an upper bound on the number of particular solutions of Equation (5). The actual number of particular solutions of Equation (5) is strictly less than this upper bound, since there are different combinations of (p_1, p_2) that produce identical particular solutions of Equation (5). For example, each of the parameter assignments (p_1, p_2) = (*0*, *0*) and (p_1, p_2) = (*0*, *b*) produces the particular solution (X_1, X_2, X_3) = (*0*, *0*, $\bar{a}$).



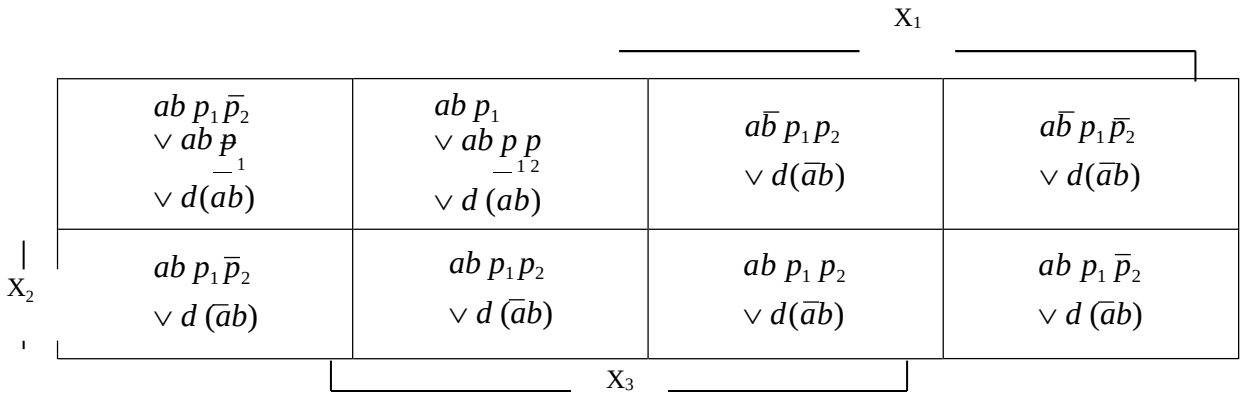
$g\,(X_1, X_2, X_3)$

Fig. (3). A natural map representation of the Boolean function g given by (5).



$g\,(X_1, X_2, X_3)$

Fig. (4). Entries of the map for the function g in Fig.(3) expanded in terms of atoms of $\mathbf{B}_{16}$ or minterms of *FB(a , b)*.



$$G\,(X_1, X_2, X_3; p_1, p_2)$$

Fig. (5). Each appearance of an entered atom in Fig.4 is ANDed with a certain element of a set of orthonormal tags, while the atom $\bar{a}b$ that appears nowhere in Fig. 4. is entered don't care.

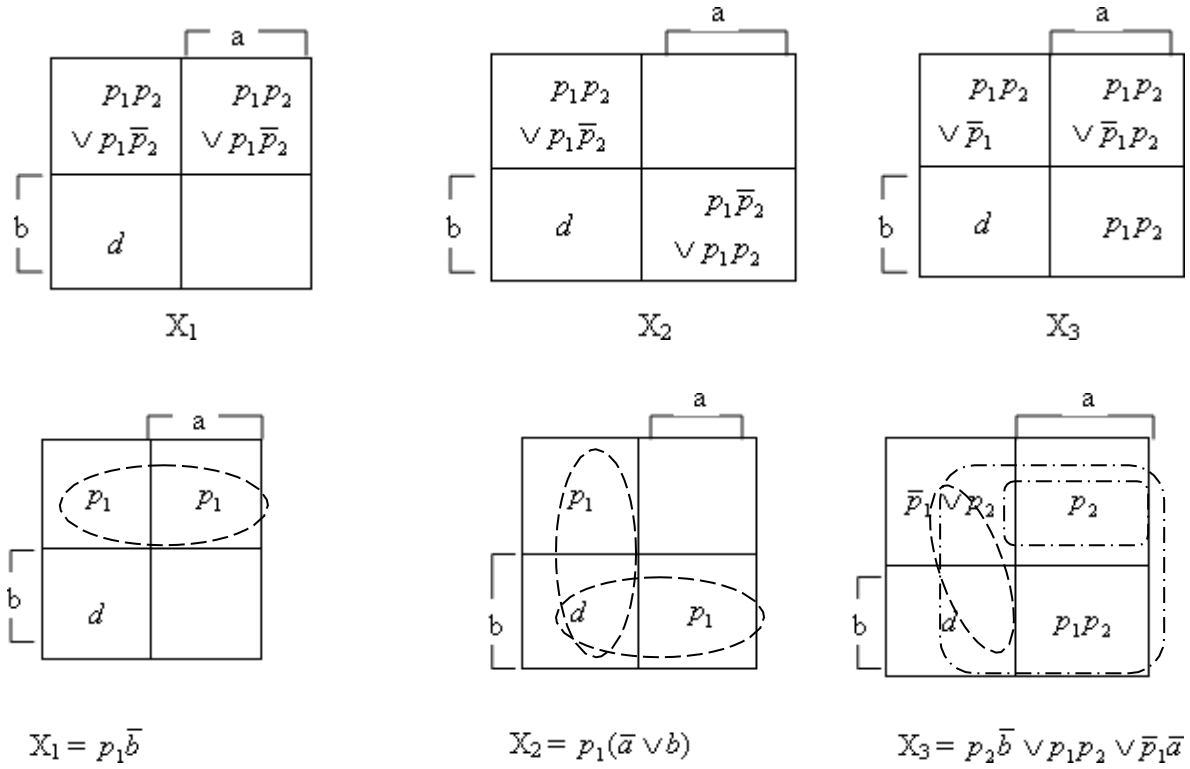


Fig. (6). VEKM expression of the parametric solution.

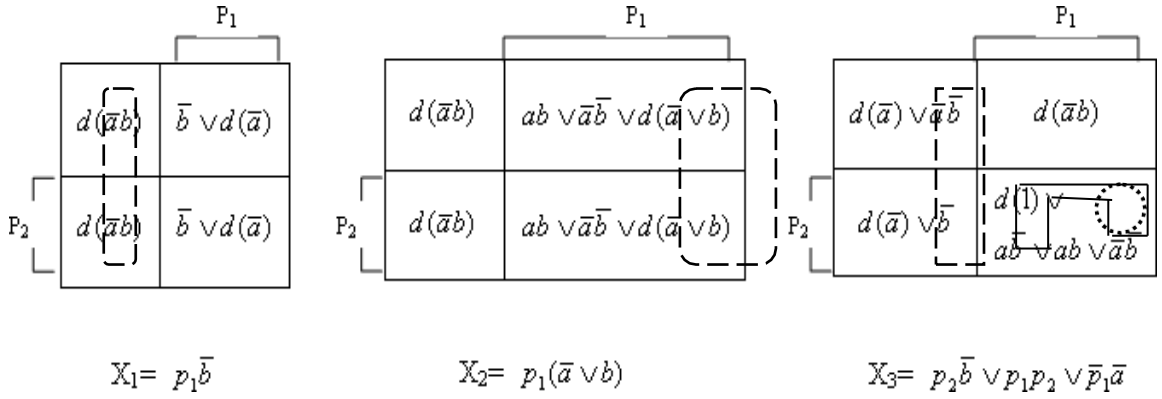


Fig. (7). An alternative VEKM expression of the parametric solution.

5. Direct Derivation of Particular Solutions

Any particular solution is obtained by randomly retaining a single instance of each of the entered atoms in the map of Fig. (3). Figure (8) shows the case where the atom ab is retained in the $\bar{X}_1\bar{X}_2\bar{X}_3$ cell, the atom ab is kept in the $\bar{X}_1X_2X_3$ cell, and the atom ab is retained in the $X_1X_2\bar{X}_3$ cell. Note that the atom $a\bar{b}$ was not entered in Fig. (3) and hence is not retained in Fig. (8).

As stated earlier, the consistency condition is $a\bar{b} = 0$, which means that the atom ab is a forbidden or cannot – happen entity, and hence it appears don't-care in each cell of Fig. (8). In the particular solution obtained, the value of the X_i variable is the disjunction of entries in the X_i half map, augmented by a don't-care $\bar{a}b$, i.e.,

$$X_1 = ab \vee d(\bar{a}b) = \bar{a}, \tag{8a}$$

$$X_2 = ab \vee ab \vee d(\bar{a}b) = \bar{a} \vee b, \tag{8b}$$

$$X_3 = ab \vee d(\bar{a}b) = b. \tag{8c}$$

The number of particular solutions for this case = 3 x 3 x 4 = 36, a number which is well below the upper bound of 64.

Brown [2, pp. 192-193] obtained a particular solution using a 5–variable Karnaugh map representation in which he used the parameters *a* and *b* as extra map variables. His solution is similar to the present one, with the exception that he did the equivalent of randomly adding the forbidden atom $\bar{a}b$ to the cell $X_1\bar{X}_2X_3$. Therefore, he obtained the particular solution

$$X_1 = ab \vee \bar{a}b = \bar{a}, \tag{9a}$$

$$X_2 = ab \vee ab, \tag{9b}$$

$$X_3 = ab \vee \bar{a}b = b, \tag{9c}$$

which is not different from (8) if the consistency condition is invoked . However, the freedom of adding $\bar{a}b$ arbitrarily asserted to one of 8 cells produced the erroneous prediction of 388 particular solutions [2, p. 192], an overestimation by a multiplicative factor of 8, and also a violation of the pertinent upper bound.

In general, the number of particular solutions for any big Boolean-algebra equation $g(\mathbf{X})=1$ can be obtained by drawing the natural Karnaugh map for $g(\mathbf{X})$ and observing the number of times each entered atom of $g(\mathbf{X})$ makes its appearance in the cells of the map, i.e.,

Number of particular solutions =

$$\Pi \text{ (Number of times an atom appears in the map of g(}\mathbf{X}\text{))}, \tag{10}$$

where the product operator (Π) in (10) runs over every entered atom in the map of $g(\mathbf{X})$.

If desired, a complete listing of all particular solutions can be obtained in a variety of ways such as (a) exhaustively and directly enumerating particular solutions by repeatedly following the strategy of this section, i. e., by noting every instance in which each entered atom is retained once in the natural map of the function *g*, (b) assigning independent values to the parameters in a parametric general solution, and (c) expanding a tree of sub-cases for a subsumptive general solution [1, 2, 8, 10].

Table (2) explicitly lists all the particular solutions of the equation g =1 in (5). These solutions are all the valid solutions (and nothing else) produced individually without any kind of overlapping or repetition. For each of these solutions g can be shown to equal $a \vee \bar{b}$ which is 1 according to the consistency condition ($\bar{a}b = 0$). However, Table (1) is not the recommended form for a solution representation since it details a large number of solutions and obscures regularities in their form.

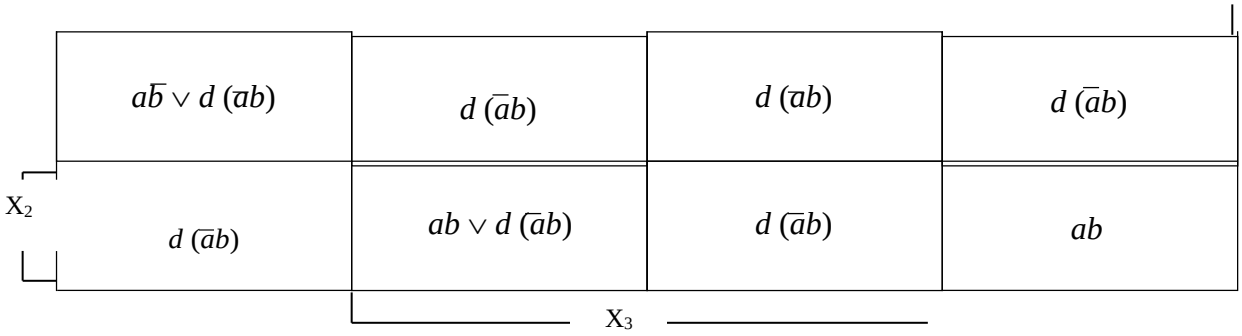


Fig.(8). Pertaining to the derivation of one particular solution.

Table (2). Listing of particular solutions of equation (5):

X_1	X_2	X_3	X_1	X_2	X_3
0	0	$\bar{a}$	$a\bar{b}$	0	$\bar{a}$
0	0	$\bar{b}$	$a\bar{b}$	0	$\bar{b}$
0	b	$\bar{a}$	$a\bar{b}$	b	$\bar{a}$
0	b	$\bar{a} \vee b$	$a\bar{b}$	b	$\bar{a} \vee b$
0	b	$\bar{b}$	$a\bar{b}$	b	$\bar{b}$
0	b	1	$a\bar{b}$	b	1
$\bar{a}$	$\bar{a}$	0	$\bar{b}$	$\bar{a}$	0
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{b}$	$\bar{a}$	$\bar{a}$
$\bar{a}$	$\bar{a}$	$a\bar{b}$	$\bar{b}$	$\bar{a}$	$a\bar{b}$
$\bar{a}$	$\bar{a}$	$\bar{b}$	$\bar{b}$	$\bar{a}$	$\bar{b}$
$\bar{a}$	$\bar{a} \vee b$	0	$\bar{b}$	$\bar{a} \vee b$	0
$\bar{a}$	$\bar{a} \vee b$	b	$\bar{b}$	$\bar{a} \vee b$	b
$\bar{a}$	$\bar{a} \vee b$	$\bar{a}$	$\bar{b}$	$\bar{a} \vee b$	$\bar{a}$
$\bar{a}$	$\bar{a} \vee b$	$\bar{a} \vee b$	$\bar{b}$	$\bar{a} \vee b$	$\bar{a} \vee b$
$\bar{a}$	$\bar{a} \vee b$	$a\bar{b}$	$\bar{b}$	$\bar{a} \vee b$	$a\bar{b}$
$\bar{a}$	$\bar{a} \vee b$	a	$\bar{b}$	$\bar{a} \vee b$	a
$\bar{a}$	$\bar{a} \vee b$	$\bar{b}$	$\bar{b}$	$\bar{a} \vee b$	$\bar{b}$
$\bar{a}$	$\bar{a} \vee b$	1	$\bar{b}$	$\bar{a} \vee b$	1

6. Conclusions

This paper presents a new method for obtaining the most compact form of the parametric general solution of a system of Boolean equations. The method is based on the use of the variable-entered Karnaugh map and hence is an effective combination of mapping and algebraic methods. The method starts by reducing a Boolean system of equations into a single equation of the form g(**X**) = 1, expressing g(**X**) in appropriate VEKM form and then modifying this VEKM by augmenting its atomic entries by appropriate elements of sets of orthonormal tags which are products of the parameters used. The method then proceeds by constructing small VEKMs for the pertinent variables **X.** Each of these latter functions is an ISBF that appears in a form suitable for VEKM minimization. The technique proposed herein is not restricted to two-valued Boolean algebras as clearly attested to by the illustrative example in Section V. As an offshoot, the paper contributes some pictorial insight on the representation of "big" Boolean algebras and functions. It also predicts the correct number of particular solutions of a Boolean equation, and produces a comprehensive list of particular solutions.

The concepts and method developed herein can be utilized in various application areas of Boolean equations [1-10, 16]. In particular, an automated version of the present Boolean-equation solver can be applied in the simulation of gate-level logic. However, such an application must handle the incompatibility between the lattice structure of 'big' Boolean algebras, which are only partially ordered, and multi-valued logics, which are totally ordered [17]. The ideas expressed herein can also be incorporated in the automated solution of large systems of Boolean equations [18]. They can also be extended to handle quadratic Boolean equations [19], Boolean ring equations [4, 5, 20], and Boolean differential equations [21].

List of Symbols

n number of input variables **X** for a Boolean function *f*(**X**).
m cardinality of the set of atoms of a Boolean algebra *B.*
k minimum number of symbols used in a free Boolean algebra to represent the elements of a Boolean carrier **B**.
A/B the set difference of sets A and B = $\{ X \mid X \in A, X \notin B\}$.
X an n-tuple ($X_1, X_2, \ldots, X_n$) of Boolean variables $X_i \in$ **B.**

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