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Spectral Response Assessment of Rugate Filters Using the Perturbation Method of Multiple Scales

Mohammed H. Bataineh

Electrical Engineering Department, Faculty of Engineering, Qassim University
P.O.Box 6677 (Email: mohbat@qec.edu.sa)

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ABSTRACT. This paper uses the perturbation method of multiple scales to obtain a closed form solution for the reflectance of rugate filters. The computational speed of the method is orders of magnitude higher than that of others. This allows studying the effect of filter's profile modifications and gives insight on the effect of the parameters involved. In addition to that, the method could be used to synthesis the spectral response of rugate filters.

Keywords: periodic structures, Optical waveguide filters, coupled-mode theory, multiple scales analysis.

1. INTRODUCTION

Rugate filters are used extensively as antireflection coatings. The usual procedure of calculating their spectral response is to utilize their stack counterpart. Thus partition the filter's profile to very thin layers and employ the *ABCD* characteristic matrix approach [1,pp.45]. As the number of partitions increases the number of multiplication required increases and thus a long computational time is needed. Methods proposed to alleviate this computation burden have been proposed by tackling the problem from periodic structure point of view. In [2], coupled-wave theory has been employed and gives results in agreement with the conventional characteristic matrix method. The drawback of applying this method is in using unclearly justified approximations. In addition to that, this method cannot easily be adapted to consider modulating the sine-wave profile with other slowly varying functions. This paper uses one of the small amplitude theories referred to as the perturbation method of multiple scales to analyze these types of filters. This is indeed one of the five different methods discussed in [3] for the analysis of mode coupling of two guided modes. The perturbation method of multiple scales was used in [4] to treat propagation of electromagnetic waves in corrugated waveguides. Its application to the present problem will not only give a fast and accurate method of evaluating and assessing the spectral response of an arbitrary refractive index profile, but also open the way forward for synthesizing the spectral response, which requires further exploration, however. In the next section, formulation of the problem in terms of the perturbation method of multiple scales is given. In Section 3, the incident and reflected interacting waves are derived. Some examples and validation of the method are presented in section 4.

2. FORMULATION

The problem being considered is the propagation of plane electromagnetic waves in a medium whose relative permittivity varies periodically with the depth of the structure, assumed to be L, as shown in Figure 1. The outer face and substrate are of refractive indices n_0 and n_s, respectively. The

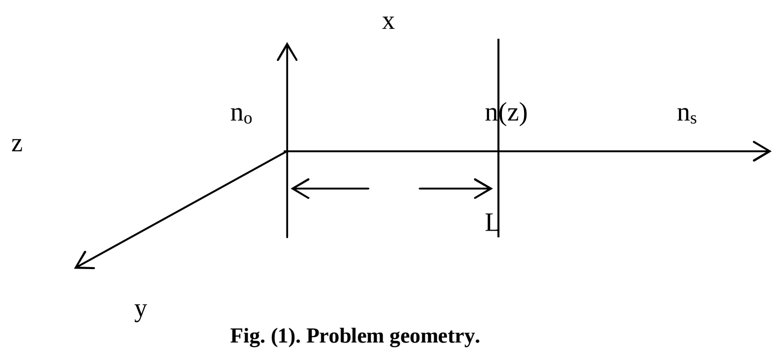

Fig. (1). Problem geometry.

The wave equation governing the electric field can be derived from Maxwell's equations and written as

$$\nabla^2 E - \nabla\, \nabla \cdot E + \omega^2 \mu_o \varepsilon_o \varepsilon_r(z)\, E = 0 \tag{1}$$

A normal incident z-directed plane wave is considered with $E = \hat{a}_x\, E_x(z)$. Substituting in Eq. (1) results in

$$\nabla^2 E_x + k_z^2 E_x = 0 \tag{2}$$

where:

$$k_z^2 = \omega^2 \mu_o \varepsilon_o \varepsilon_r(z)$$

or

$$k_z = \sqrt{\omega^2 \mu_o \varepsilon_o}\ \sqrt{\varepsilon_r(z)} = \tilde{k}_o\, n(z) \tag{3}$$

Here, n(z) is the refractive index of the rugate filter considered here to take a general form

$$n(z) = n_a\,(1 + \delta \sin(K_1 z))$$

Where n_a is the average unperturbed refractive index, δ is the perturbation index and K_1 is the wave number of the refractive index profile given by

$$K_1 = \frac{4\pi}{\lambda_1} n_a \tag{4}$$

λ_1 is the designed wavelength or wavelength at which peak performance is expected. Now k_z in Eq. (3) can be casted in the following form

$$k_z = \tilde{k}_o n_a\,(1 + \delta \sin(K_1 z))$$
$$= k_o\,(1 + \delta \sin(K_1 z)) \tag{5}$$

The governing equation (2) is a homogenous second order differential equation with variable coefficients whose solution is sought via the perturbation method of multiple scales.

3. WAVE INTERACTION EQUATIONS

The reflectivity or reflectance of a periodically varying refractive index profile is due to an interaction between incident and reflected waves. The derivation here is presented by expanding the field in powers of δ in the form

$$E_x(z) = E_x^{(0)}(z_o, z_1) + \delta E_x^{(1)}(z_o, z_1) + \ldots \tag{6}$$

Where $z_o = z$ is a fast varying scale, and $z_1 = \delta z_o$ is a slowly varying scale. Expressing the derivatives in terms of z_o and z_1 by using the chain rule and substituting in Eq. (2), then equating coefficients of equal powers of δ to obtain

$$\underline{O\,(\delta^0)}$$

$$\frac{\partial^2 E_x^{(0)}}{\partial z_o^2} + k_o^2 E_x^{(0)} = 0 \tag{7}$$

$$\underline{O\,(\delta^1)}$$

$$\frac{\partial^2 E_x^{(1)}}{\partial z_o^2} + k_o^2 E_x^{(1)} = -2\frac{\partial^2 E_x^{(0)}}{\partial z_o \partial z_1} - k_o^2 \sin(K_1 z_o) E_x^{(0)} \tag{8}$$

Note that Eq. (7) corresponds to unperturbed system whose general solution is given by

$$E_x^{(0)}(z_o, z_1) = A(z_1)\, e^{-j k_o z_o} + B(z_1)\, e^{j k_o z_o} \tag{9}$$

Eq. (9) suggests propagation of two contra-directional traveling waves. The functions $A(z_1)$ and $B(z_1)$ are slowly varying functions representing the amplitudes of the incident and reflected waves, respectively. They are determined from the solvability conditions of the first order problem.

By substituting Eq. (9) in (8), we obtain

$$\frac{\partial^2 E_x^{(1)}}{\partial z_o^2} + k_o^2 E_x^{(1)} = j\,2\,k_o\left(A'(z_1)\, e^{-j k_o z_o} - B'(z_1)\, e^{j k_o z_o}\right) - \frac{k_o^2}{2j}\left(e^{j K_1 z_o} - e^{-j K_1 z_o}\right)\left(A(z_1)\, e^{-j k_o z_o} + B(z_1)\, e^{j k_o z_o}\right) \tag{10}$$

The primes denote differentiation with respect to z_1. Had we attempted a straightforward expansion (corresponding to $\frac{\partial}{\partial z_1} = 0$), we would found that it breaks down when the following resonance condition is satisfied

$$2\,k_o = K_1 \tag{11}$$

This is what is usually referred to as the Bragg condition. However, it is not necessary that this condition be valid exactly, it is sufficient if the phase $\pm\, j\,(2\,k_o - K_1)$ has a slow spatial variation. The nearness to resonance is measured by a detuning parameter α, defined by

$$2\,k_o - K_1 = \delta\,\alpha \tag{12}$$

Finding a particular solution of Eq. (10) would lead to secular terms; *i.e.*, solutions that are proportional to $z_0 e^{\pm j k_0 z_0}$. This means that $E_x^{(1)}$ would soon become greater than $E_x^{(0)}$ and, consequently, the perturbation expansion in Eq. (6) for E_x is not uniform and it breaks down. To eliminate the secular producing terms from the right-hand side of Eq. (10), we set the coefficients of $e^{\pm j k_0 z_0}$ equal to zero and arrive at the following solvability conditions for the first-order problem

Invoking the solvability conditions in Eq. (9) requires

$$j\,2\,k_o\,A'(z_1) + \frac{k_o^2}{2j} B\,e^{\,j\,(2k_o - K_1)\,z_o} = 0 \tag{13}$$

or

$$A'(z_1) = \frac{k_o}{4}\,B\,e^{j\alpha z_1} \tag{14}$$

and

$$-j\,2\,k_o\,B'(z_1) - \frac{k_o^2}{2j} A\,e^{-j\,(2k_o - K_1)\,z_o} = 0 \tag{15}$$

or

$$B'(z_1) = \frac{k_o}{4}\,A\,e^{-j\alpha z_1} \tag{16}$$

Equations (14) and (16) are first order coupled equations with $\frac{k_o}{4}$ as a coupling coefficient. They can be solved using any standard differential equations solver to determine the amplitude reflection coefficient r , which is the ratio of the backward-propagating to the forward-propagating amplitude. However, to obtain a closed form formula for the reflectance, these two equations are decoupled by differentiating equation (14) and substituting from Eq. (16). The result is a second order differential equation given by:

$$A'' - j\,\alpha\,A' - \left(\frac{k_o}{4}\right)^2 A = 0 \tag{17}$$

Whose solution is given by

$$A(z_1) = e^{\,j\frac{\alpha}{2} z_1}\left(C_1\,e^{-\beta z_1} + C_2\,e^{\beta z_1}\right) \tag{18}$$

β in equation (18) is given by

$$\beta = \sqrt{\left(\frac{k_o}{4}\right)^2 - \left(\frac{\alpha}{2}\right)^2} \tag{19}$$

The constants C_1 and C_2 are determined from the conditions imposed at the boundaries of the filter. The solution for $B(z_1)$ can be obtained from Eq. (14), and may be written as

$$B(z_1) = \frac{4}{k_o}\left[-C_1\left(\beta - j\frac{\alpha}{2}\right)e^{-\beta z_1} + C_2\left(\beta + j\frac{\alpha}{2}\right)e^{\beta z_1}\right]e^{-j\frac{\alpha}{2} z_1} \tag{20}$$

Actually there is no need to impose two conditions to evaluate C_1 and C_2 since the amplitude reflection coefficient is given by the ratio of the backward propagating wave to the forward propagating wave of Eq. (9). To find the ratio of these two constants, we take the condition at the filter-substrate interface at z_o = L. The reflection coefficient, as defined above, when evaluated at z_o= L may be written also in terms of filter's refractive index profile at z_o= L and the that of substrate layer, i.e.,

$$r\Big|_{z_o=L} = r_s = \frac{n(L)-n_s}{n(L)+n_s} = \frac{B(\delta L)}{A(\delta L)}\, e^{j\,2\,k_o L} \tag{21}$$

Upon using Eqs.(18) and (20) into Eq. (21) one can obtain

$$\frac{C_2}{C_1} = \frac{\left(\frac{k_o}{4}\, r_s + (\beta - j\frac{\alpha}{2})\, e^{j K_1 L}\right)}{\left(-\frac{k_o}{4}\, r_s + (\beta + j\frac{\alpha}{2})\, e^{j K_1 L}\right)}\, e^{-2\beta\delta L} \tag{22}$$

The amplitude reflectivity at any point down the structure can now be written as:

$$r(z_0) = \frac{B(\delta z_0)}{A(\delta z_0)}\, e^{j\,2\,k_o z_0} \tag{23}$$

After some manipulation, Eq.(23) can be written in the following form

$$r(z_0) = \frac{-\left(\beta - j\frac{\alpha}{2}\right) + \frac{C_2}{C_1}\left(\beta + j\frac{\alpha}{2}\right) e^{2\beta\delta L}}{\frac{k_o}{4}\left(1 + \frac{C_2}{C_1}\, e^{2\beta\delta L}\right)}\, e^{j K_1 z_0} \tag{24}$$

The recursive Fresnel reflection formula [4] could now be used to find the reflection coefficient, ρ at z = 0:

$$\rho = \frac{r_a + r\big|_{z_o=0}}{1 + r_a\, r\big|_{z_o=0}} \tag{25}$$

Where r_a is the reflection coefficient at the outer interface, given by

$$r_a = \frac{n_0 - n(0)}{n_0 + n(0)} \tag{26}$$

The power reflection coefficient, or reflectance, is given by

$$R = \rho\rho^* \qquad (27)$$

Various variations of the refractive index profile can be assessed by evaluating the reflectance, as defined in Eq.(27), with respect to the wavelength.

4. ILLUSTRATIVE EXAMPLES

To validate the method presented in this paper we use the same example given in [2]. The parameters used are 100 – cycle of rugate with $\delta = 0.05$, $\lambda_1 = 0.55$ µm, $n_s = 1.52$, and $n_o = 1$. The spectral response of power reflection coefficient, R, versus the wavelength, λ is shown in Fig. 2. The result is in excellent agreement with that given in [2] obtained in much less time when compared on the same machine. This example corresponds to uniform corrugation of rugate filter. Nonuniform corrugations could also be used. A variety of nonuniform perturbations can be studied and handled by the method. A notch–like response may be obtained by inverting the filter's profile at its mid-point. This is achieved by letting δ to take the form:

$$\delta(z_1) = \begin{cases} \delta_o & 0 \le z_1 \le \dfrac{L}{2} \\ -\delta_o & \dfrac{L}{2} \le z_1 \le L \end{cases} \qquad (28)$$

The effect of this is shown in Fig. 3. Using the profile given in Eq. (28) means that a phase reversal at the middle of the structure has been made. This opens a narrow sharp transmission at the resonance wavelength. This is look a lot like the difference mode radiation pattern obtained in antenna arrays [6,pp.245.].

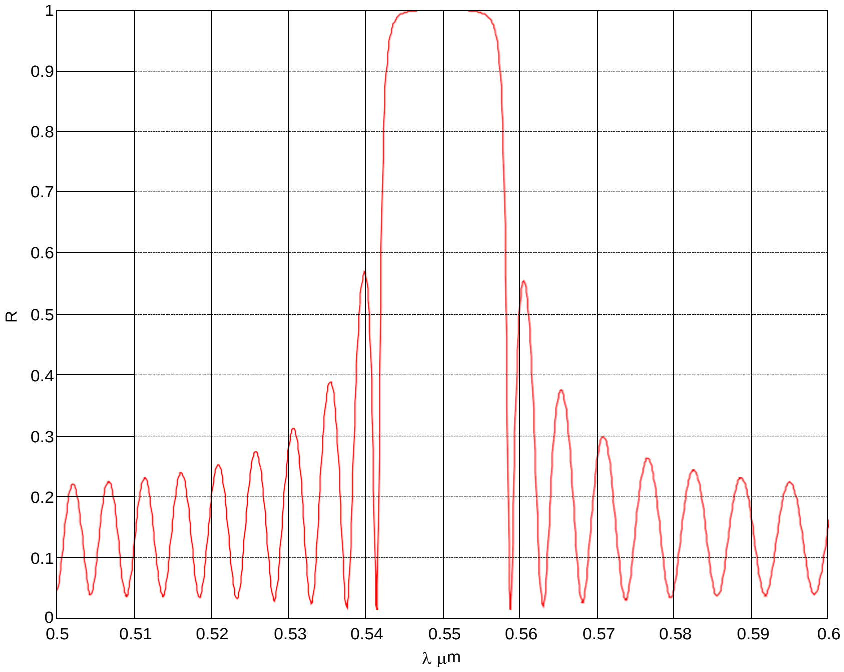

Fig. (2). Spectral response of uniform rugate.

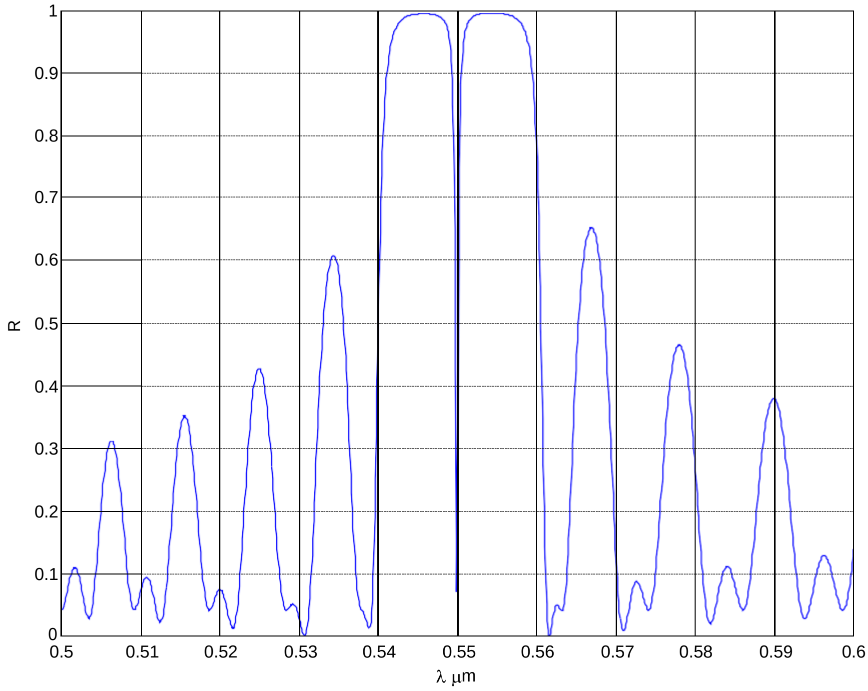

Fig. (3). Notch-like spectral response of nonuniform rugate.

5. CONCLUSIONS

Perturbation method of multiple scales can be used with arbitrary filter's profile to assess the response of rugate filters. In addition to its computational speed the method offers an alternative platform for synthesizing rugate filters. Important applications, notably wave division multiplexing, immerged by obtaining spectral response to suite the required application.

6. REFERENCES

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تقييم الاستجابة الطيفية للمرشحات روجيت باستخدام أسلوب من قلقلة ذات مستويات متعددة

محمد حسين البطاينه

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mohbat@qec.edu.sa

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