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A Comparison of Algebraic and Map Methods for Solving General Boolean Equations

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ABSTRACT. This paper identifies the main types of solutions of Boolean equations as subsumptive general solutions, parametric general solutions and particular solutions. The paper offers a tutorial exposition, review, and comparison of the three types of solutions by way of two illustrative examples solved by both map and algebraic techniques. Map techniques are demonstrated to be at least competitive with (and occasionally superior to) algebraic techniques, since they have a better control on the minimality of the pertinent function representations, and hence are more capable of producing more compact general parametric and subsumptive solutions.

Key words**:** Solution of Boolean equations, Subsumptive general solutions, Parametric general solutions, Particular solutions, Map techniques, Algebraic techniques.

1. Introduction

The topic of Boolean equations has been a hot topic of research for almost two centuries and its current importance can be hardly overestimated. Boolean-equation solving permeates many diverse areas of modern science such as biology, grammars, chemistry, law, medicine, spectrography, and graph theory [1]. It is also an indispensable tool in operations research [2], the cryptanalysis and breaking of ciphers [3], Boolean satisfiability (SAT) problem solving [4], the synthesis, simulation and testing of digital networks and VLSI systems [5, 6], output encoding and state assignments of finite state machines [7], and automatic test-pattern generation [8].

There is a huge number of methods for solving Boolean equations, covering the general case of big Boolean equations, or the special case of bivalent, switching or truth equations (See, e.g., [1-3, 5, 9-26]). Most prominent among these methods are the two important classes of algebraic methods and tabular or map methods. The main types of solutions of Boolean equations can be identified as subsumptive general solutions, parametric general solutions and particular solutions. In a subsumptive general solution, each of the variables is expressed as an interval based on successive conjunctive or disjunctive eliminants of the original function. In a parametric general solution, each of the variables is expressed via arbitrary parameters, i.e., via freely chosen elements of the underlying Boolean algebra. A particular solution is an assignment from the underlying Boolean algebra to every pertinent variable that makes the Boolean equation an identity.

This paper is a tutorial exposition, review, and comparison of the use of algebraic methods and map methods in obtaining the main types of solutions of Boolean equations. We will consider two algebraic methods, both due to Rudeanu [9, 22], and a map method that does not rely on the use of the classical Karnaugh map (CKM) but on the use of the variable-entered map [20, 23, 25-30]. Though the variable-entered Karnaugh map (VEKM) is typically classified among (and used herein as a representative of) map methods, it is not really a purely-map method, but it is semi-algebraic in nature. The VEKM is the natural map for representing finite big Boolean functions that are not necessarily two-valued functions [26]. A Boolean function of n variables has 2^n VEKM representations (depending on the choice of map and entered variables) ranging from a CKM (n map variables and 0 entered variables), and a purely-algebraic expression (0 map variables and n entered variables). The VEKM methods therefore include purely-algebraic methods as a special case. Hence, they can always take full advantage of the results provided by the algebraic theory. Moreover, they have a better control on the minimality of pertinent function representation.

The rest of the paper is organized as follows. Section 2 views the first algebraic method of Rudeanu presented in his pioneering and seminal text [9], while Section 3 discusses use of the VEKM in obtaining a subsumptive general solution of Boolean equations. Section 4 studies the second algebraic method of Rudeanu presented in his paper [22], while Section 5 assesses it in terms of the results of Section 3. The second algebraic method of Rudeanu in [22] is shown to secure minimality over a set of chosen coefficients and not over the more basic set of pertinent variables and free generators. Therefore, this method is not always as efficient as the VEKM method. Section 6 adds a discussion of using the VEKM in obtaining a general parametric solution of Boolean equations. In Sections 3, 4, and 6, we offer a tutorial exposition of the subject by way of two illustrative examples that produce compact general solutions and then expand them to particular solutions. Section 7 concludes the paper.

2. First Algebraic Method of Rudeanu

In this section, we review the classical technique of constructing subsumptive general solutions for a Boolean system of equations. More details can be found in [1, 20, 23], and a formal proof is available in [9]. To distinguish this technique from that in [22], we call it the first algebraic method of Rudeanu, while the technique in [22] is labeled as the second algebraic technique of Rudeanu.

An n-variable Boolean system on a Boolean algebra B is a set of k simultaneously asserted equations. This system is equivalent to the single equation

$$f(\mathbf{X}) = 0, \tag{1}$$

where $\mathbf{X} = [X_1, X_2, \ldots, X_n]^T$ is a vector of n components X_i each belonging to the Boolean carrier B. The subsumptive solution is obtained by constructing the eliminants

$$f_n(X_1, X_2, \ldots, X_n), \ldots, f_i(X_1, X_2, \ldots, X_{i-1}, X_i), \ldots, f_2(X_1, X_2), f_1(X_1), f_0$$

by setting $f_n = f$ and using the recursion

$$f_i(X_1, X_2, \ldots, X_{i-1}) = (f_i / \;\; _i) \wedge (f_i / X_i), \;\; i = n, n-1, \ldots, 1. \tag{2}$$

Note that $f_{i-1}$is the conjunctive eliminant of f_i with respect to the singleton $\{X_i\}$ [1]. This means that f_{i-1} is a conjunction of the two ratios, subfunctions, or restrictions

$$f_i \;\;\; _i = f_i(X_1, X_2, \ldots, X_{i-1}, 0), \tag{3}$$

$$f_i / X_i = f_i(X_1, X_2, \ldots, X_{i-1}, 1), \tag{4}$$

obtained from f_i by setting or restricting X_i in it to 0 and to 1, respectively. For short, these two ratios will be denoted by f_i (0) and f_i (l), respectively.

The classical method for producing a subsumptive general solution is by successive elimination of variables, a technique transforming the problem (1) of solving a single equation of n variables to that of solving n equations of one variable each. The solution requires a separate consistency condition.

$$f_0 = 0, \tag{5a}$$

plus expressing each of the pertinent variables as an interval of functions ofthe preceding variables, namely:

$$s_i(X_1, X_2, \ldots, X_{i-1}) \le X_i \le t_i(X_1, X_2, \ldots, X_{i-1}), \quad i = 1, 2, \ldots, n. \tag{5b}$$

where the s_i and t_i functions can be expressed as completely specified Boolean functions, namely

$$s_i = f_i(0) \tag{6a}$$

$$t_i = \ {}_i\,(1) \tag{6b}$$

The form of the general solution above allows all the particular solutions of (1), and nothing else, to be generated as a tree. Since the method of this section is superseded by the second algebraic method of Rudeanu [22], we will not discuss it any further. The examples on this method available in [1] demonstrate that this technique is not only tedious, but it also fails to produce compact solutions. However, it was necessary to introduce this technique herein since it is the basis of the improved techniques in Sections 3 and 4.

3. VEKM Subsumptive Solution

To leave room for further simplification, the s_i and t_i functions in (5b) are expressed as incompletely specified Boolean functions (ISBFs) in the interval form [1]

$$f_i(0)\ \ {}_i\,(1) \le s_i \le f_i(0), \tag{7a}$$

$$ {}_i\,(1) \le t_i \le f_i(0)\ \ {}_i\,(1). \tag{7b}$$

Now, these expressions can be adapted for VEKM manipulation by converting them into the incompletely-specified or don't-care expressions [20, 23]

$$s_i = f_i(0)\ {}_i\,(1)\ \ d(f_i(0)), \tag{7c}$$

$$t_i = \ {}_i\,(1)\ \ d(f_i(0)). \tag{7d}$$

The VEKM is well suited for a divide-and-conquer implementation of the complementation, ANDing, ORing and minimization operations needed in (7c) and (7d). The procedure is well illustrated by the following two examples.

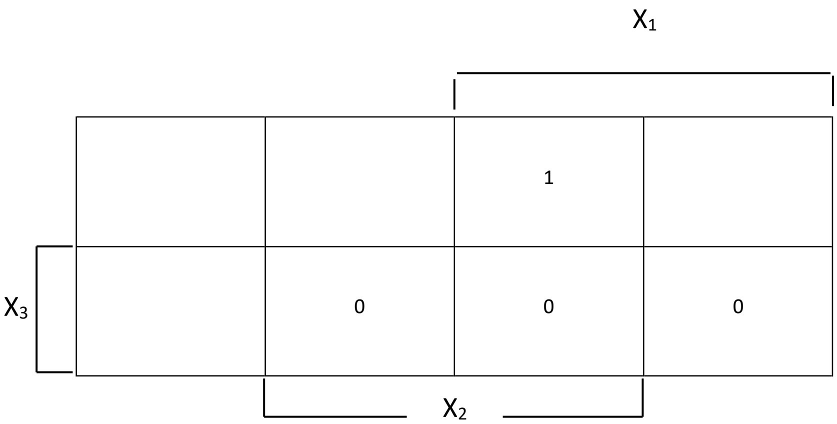
Example 1:

Let the function $f(X_1, X_2, X_3)$:$\mathbf{B}_4^3 \rightarrow \mathbf{B}_4$ which satisfies (1) be given by

$$f(x_1, x_2, x_3 \; _1 \; _2 \qquad _3 \; X_1 X_2 \; _3 \, , \tag{8}$$

This function is represented by the VEKM of Fig. (1), which actually serves as a natural map for Boolean functions over $\mathbf{B}_4$. The detailed VEKM subsumptive solution is obtained via the VEKMs in Fig. (2). The final subsumptive solution is given by the compact form:

$$\begin{aligned} {}_1X_2 \;\; &\leq \;\; {}_3 \leq 1, \\ {}_1 &\leq \;\; {}_2 \leq 1, \\ 0 \leq {}_1 &\leq 1, \\ 0 &= 0. \end{aligned} \tag{9}$$



$f(X_{1,}, X_2, \; X_3)$

Fig. (1). A VEKM representation of the Boolean function $f_3 = f(X_1, X_2, X_3)$ of Example 1, as expressed by equation (8).

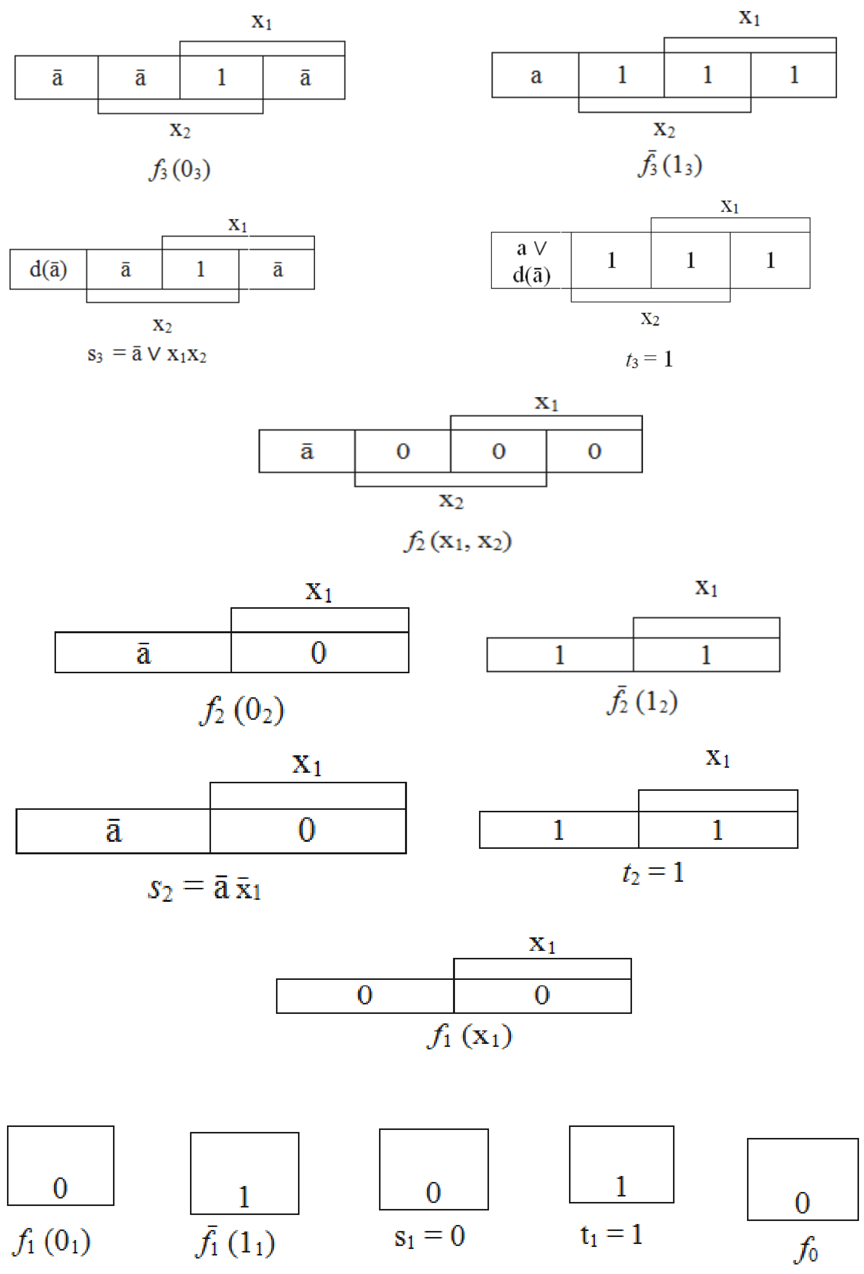


Fig. (2). Steps of the VEKM subsumptive solution for Example 1.

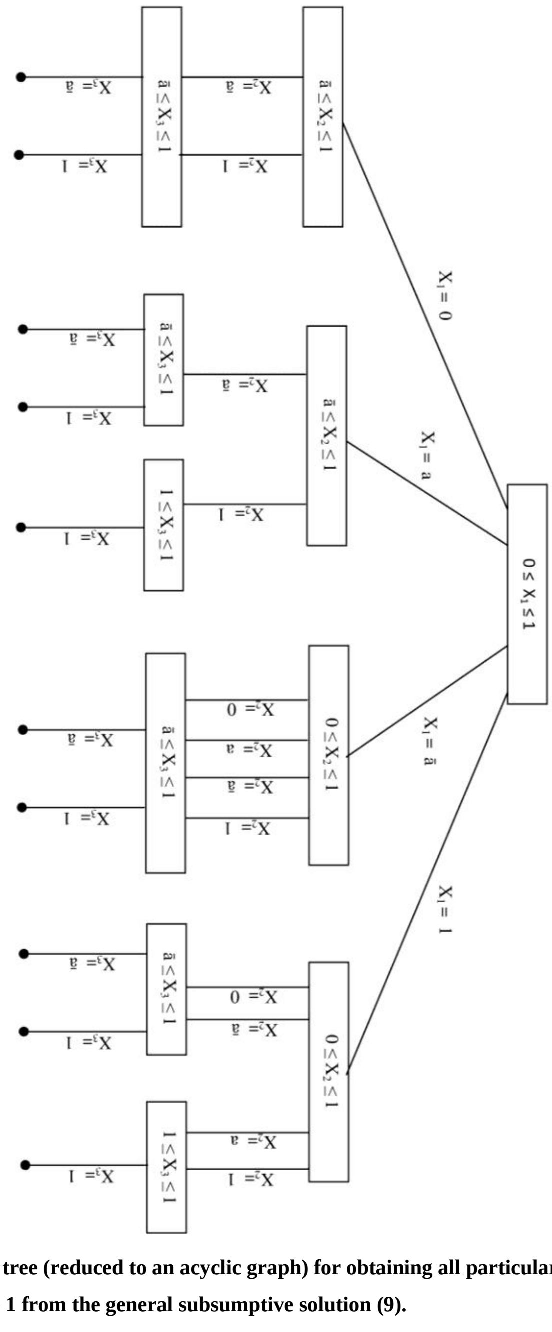


Fig. (3). Expansion tree (reduced to an acyclic graph) for obtaining all particular solution of Example 1 from the general subsumptive solution (9).

A list of all particular solutions is neither compact nor insightful as a general solution. Such a listing is produced via expansion trees from the general solutions. Figure (3) shows the expansion tree used in producing all 21 particular solutions of (1) for $f = 0$ from the general subsumptive solution (9). To save space, we combined common nodes in the tree, thereby reducing it to an acyclic graph.

Example 2:

The function $f(X_1, X_2, X_3)$:$\mathbf{B}^3_{16} \rightarrow \mathbf{B}_{16}$given by

$$f(X_1, X_2, X_3) = ab \quad {}_3 \quad {}_1X_3 \quad X_2X_3 \quad {}_3 \quad {}_2 \ {}_3 \quad {}_1 \ {}_3\,, \tag{10}$$

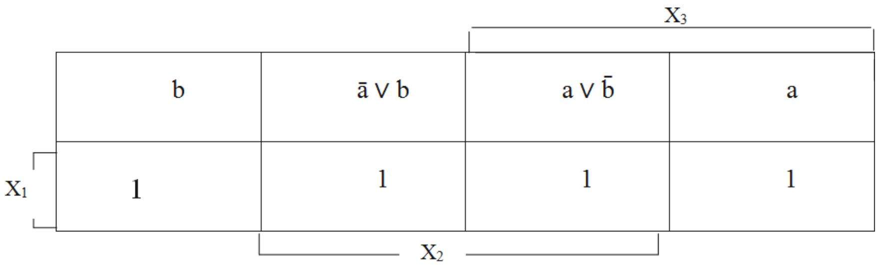
is represented by the VEKM of Fig. (4), which actually serves as a natural map for Boolean functions over B_{16}. The detailed solution is obtained via the VEKM in Fig.(5). The final subsumptive solution is:

$$b \leq {}_3 \leq \,,$$

$$0 \leq {}_2 \leq a\ b\,,$$

$$0 \leq {}_1 \leq 0,$$

$$ab = 0. \tag{11}$$



f (X1, X2, X3)

Fig. (4). A VEKM representation of the Boolean function $f_3 = f\,(X_1, X_2, X_3)$ of Example 2.

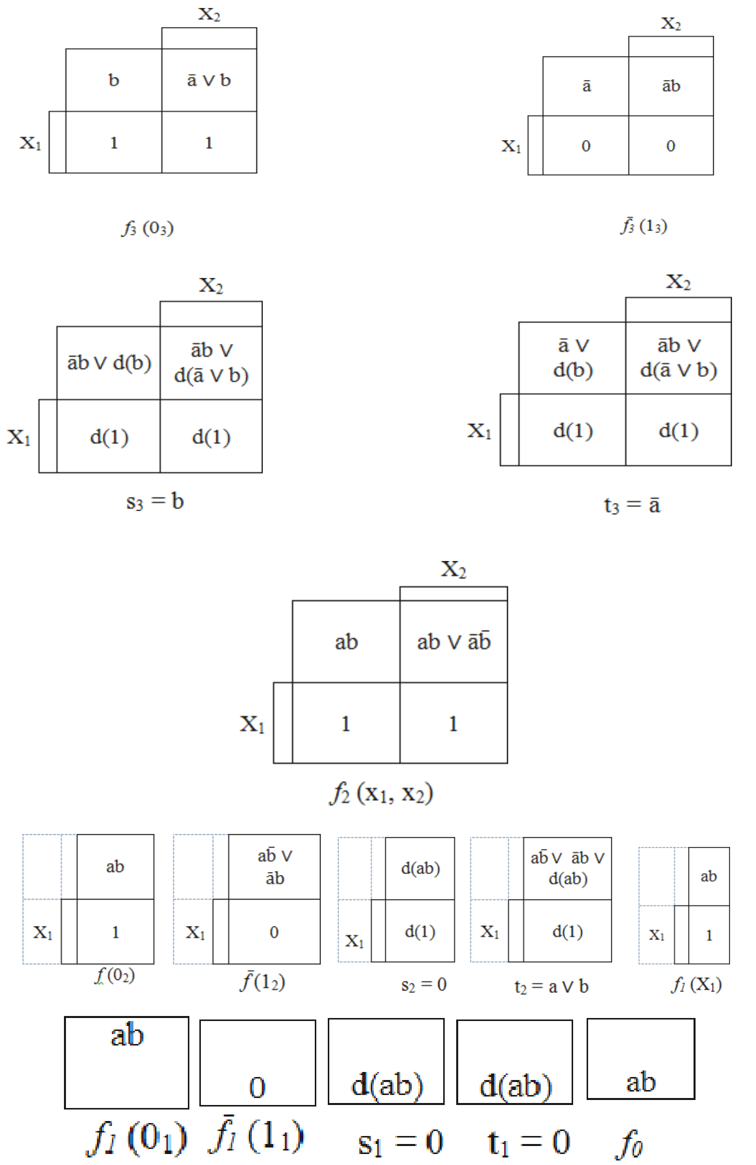


Fig. (5). Steps of the VEKM subsumptive solution for Example 2.

Figure (6) illustrates the acyclic-graph production of all 8 particular solutions of equation (11) for f = 0 from the general solution (11). Here, the consistency condition (ab = 0) made the underlying Boolean algebra collapse from the hypercube lattice of $\mathbf{B}_{16}$ in Fig. (7) to the cubic lattice of $\mathbf{B}_8$ in Fig. (8) [23, 26].

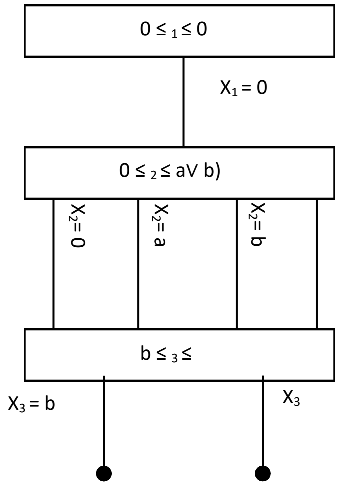


Fig. (6). Expansion tree (reduced to an acyclic graph) for obtaining all the particular solutions of Example 2 from the general subsumptive solution (11).

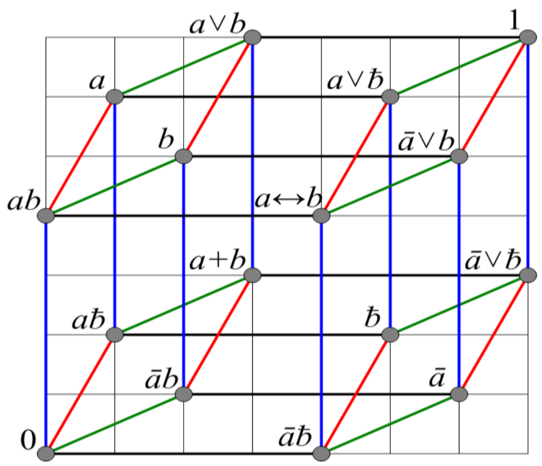


Fig. (7). A hypercube lattice indicating the partial ordering among the 16 elements of B_{16}.

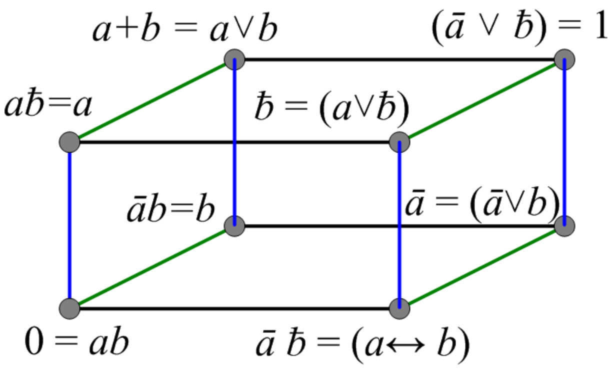


Fig. (8). The lattice in Fig. (7) when collapsed under the condition ab = 0.

4. Second Algebraic Method of Rudeanu

Rudeanu [22] proposed a second algebraic method for solving the Boolean equation $f(\mathbf{X}) = 0$. The function $f(\mathbf{X}) = fn(\mathbf{X})$ of n variables is written as

$$fn(\mathbf{X}) = \quad \vee \quad \bar{} \quad \vee \tag{12}$$

where the coefficients , , are functions of the (n-1) remaining variables $(\boldsymbol{X}/X_n) = (X_1, X_2, ..., X_{n-1})$. The subsumptive solution for in terms of the other (n-1) variables is provided by the double inequality

$$\bar{} \;, \tag{13}$$

provided the following consistency condition is satisfied

$$f_{n-1}(\mathbf{X}/\quad) = \quad \vee \tag{14}$$

Thus the solution of (12) is provided partially by (13) for , and is reduced to the solution of (14) for the remaining (n-1) variables. The iteration of the above procedure leads to a successive elimination of variables and the production of a subsumptive solution for each variable in terms of the earlier variables, in addition to a final consistency condition that involves no variables but involves constants of the underlying Boolean Algebra.

Example 1 (revisited):

We apply the iterative procedure (12)-(14) to the function in (8)(of Example 1) to obtain

$$f = f_3(X_1, X_2, X_3) \quad {}_1 \; {}_2 \quad {}_3\, X_1X_2 \; {}_3 = A_3\, X_3\, B_3 \; {}_3\; C_3, \tag{15a}$$

$$C_3 = {}_1\,{}_2\,,\; A_3 = 0,\; B_3\; {}_1X_2,$$

$$(\quad X_1X_2 \ \leq \ X_3 \leq 1, \tag{16a}$$

$$f_2(X_1, X_2)= A_3B_3 \ C_3 \qquad _1 \ \ _2 = A_2\, X_2 \ B_2 \quad _2 \ C_2, \tag{15b}$$

$$B_2 \qquad _1, \ A_2 = C_2 = 0,$$

$$_1 \leq X_2 \leq 1, \tag{16b}$$

$$f_1(X_1)=A_2B_2C_2 = 0 = A_1X_1B_1 \quad _1C_1, \tag{15c}$$

$$A_1 = B_1= C_1= 0,$$

$$0 \leq X_1 \leq 1,(16c)$$

$$f_0= A_1B_1\, C_1 = 0$$

$$0 = 0 \tag{16d}$$

Equations (16a)-(16c) constitute the subsumptive solution, while equation (16d) is the final consistency condition. These equations are exactly the ones in (9). Here, the second algebraic method of Rudeanu produces the same solution as the VEKM technique.

Example 2 (revisited):

We apply the iterative procedure (12)-(14) to the function in equation (10) (of Example 2) to obtain

$$f_3(X_1, X_2, X_3 \ ab \ _1 \ b \ _2) \ X_3 \ b \ _2b \ _1 \ _3ab$$

$$= A_3X_3 \ B_3 \quad _3 \ C_3, \tag{17a}$$

$$A_3 \ \ ab \ \ _1 \ \ b \ \ _2, B_3 \ \ b \ \ _2 \ \ b \ \ _1, C_3 = \ ab,$$

$$_3 \ b \ a \ _1 \ b \ _2 \ b \ _2 \ a \ _1 \ ab \ \ b \ \ _1 \ \ _1 \ _2.$$

Hence, the solution for X_3 is

$$B_3 \leq \quad _3 \leq \ _3,$$

$$b \qquad _2 \ \ b \quad _1 \ \leq \quad _3 \leq \ ab \ \ b \quad _1 \qquad _1 \quad _2), \tag{18a}$$

with the consistency condition

$$f_2(X_1, X_2) = A_3 \ B_3 \ C_3$$

$$ab \qquad _1 \, b \quad _2 \quad b \qquad _2 \quad b \quad _1) \ ab$$

$$ab \ \ _1 \ \ b \ \ _1 \ \ _1X_2 \ \ b \ \ _1 \ \ b \quad _2 \, b \quad _1X_2 \ ab$$

$$_1 \ \ b \ \ b \quad _1) \ X_2 \ \ ab \quad _1 \ \ b \ \ _1 \quad b \quad _1 \ ab$$

$$_1 \quad b \quad b \quad _1) \ X_2 \qquad b \qquad _1 \ ab$$

$$= A_2X_2 \ \ B_2 \quad _2 \ \ C_3, \tag{17b}$$

$$A_2 \ \ _1 \ \ b \ \ b \ \ _1, \ \ B_2 = 0, \quad C_2 \quad b \quad _1ab,$$

$$_2 \ a \ _1 \ a \ b \ b \ _1 \ _1 \ ab \ a \ b \ a \ _1 \ b \ _1 \ ab.$$

Hence, the solution for X_2 is

$$B_2 \leq {}_2 \leq {}_2,$$

$$0 \leq {}_2 \leq a\ {}_1\ b\ {}_1\ ab, (18b)$$

with the consistency condition

$$f_1(X_1) = A_2\ B_2\ \ C_2\ \ \ \ b\ \ \ \ {}_1\ \ ab = A_1X_1\ \ B_1\ \ {}_1\ C_1, (17c)$$

$$A_1\ b\ ,\ {}_1 = 0,\ C_1 = ab,$$

$${}_1 = ab,$$

Hence, the solution for X_1 is

$$B_1 \leq {}_1 \leq {}_1,$$

$$0\ \leq\ \ {}_1 \leq\ ab, \tag{18c}$$

with the consistency condition

$$f_0 = A_1\ B_1\ C_1 = ab = 0, \tag{18d}$$

Relations (18a)-(18d) can be combined to give the subsumptive solution:

$$b\ \ \ {}_2\ b\ \ {}_1\ \leq\ \ {}_3\ \leq\ \ ab\ \ b\ \ {}_1\ \ \ \ {}_1\ {}_2)$$

$$0 \leq {}_2 \leq a\ {}_1\ b\ {}_1\ ab)$$

$$0\ \leq\ \ \ {}_1\ \leq\ ab$$

$$ab = 0. \tag{19}$$

As an afterthought, the solution (19) can be refined by applying the condition (ab = 0) to the preceding double inequalities. A result of this step is that ($0 \leq X_1 \leq 0$), which means that ($X_1 = 0$); a condition that can be applied to the inequalities of X_3 and X_2 to obtain: $b\ {}_2 \leq {}_3 \leq b\ \ {}_2)$

$$0 \leq\ \ {}_2\ \ \leq\ a\ \ b$$

$$0 \leq\ \ {}_1\ \leq 0$$

$$ab = 0 \tag{20}$$

But still even after this simplification, the solution (20) is still less compact than its equivalent one in (11). The particular – solution tree generated by it (see Fig. 9) is more involved and tedious to produce than the corresponding one in Fig.(6).

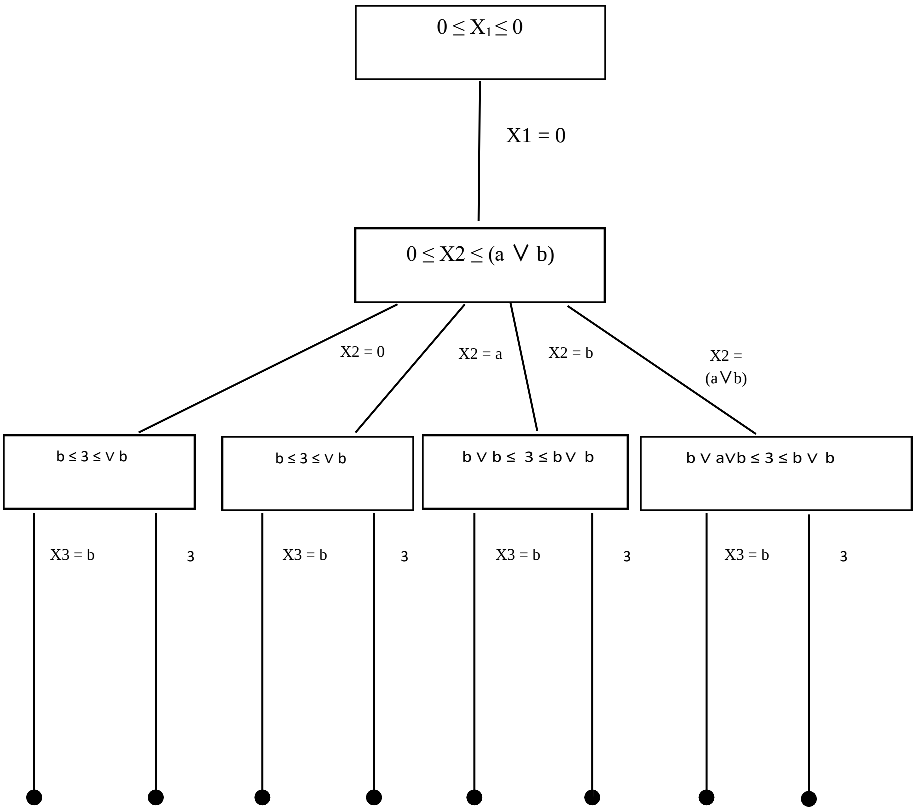


Fig. (9). Expansion tree for obtaining all the particular solutions of Example 2 from the general subsumptive solution (20).

5. An Interpretation of Rudeanu Second Method

Rudeanu Second Method produces a sequence of equationsf_i = 0, (i = n down to i = 0), wherethe function f_i= $f_i(X_1, X_2, ..., X_{i-1}, X_i)$ is

$$f_i \; = A_i \, X_i \;\; B_i \quad {}_i \;\; C_i = (\, A_i \;\; C_i \,) \, X_i \, (\, B_i \;\; C_i \qquad {}_i, \tag{21}$$

The function f_i can be represented by the VEKM in Fig. (10) in which the subfunctions$f_i(1_i)$ and $f_i(0_i)$ are:

$$f_i(1_i) = f_i(X_1, X_2, ..., X_{i-1}, 1) = A_i \;\; C_i, \tag{22}$$

$$f_i(0_i) = f_i(X_1, X_2, ..., X_{i-1}, 0) = B_i \;\; C_i, \tag{23}$$

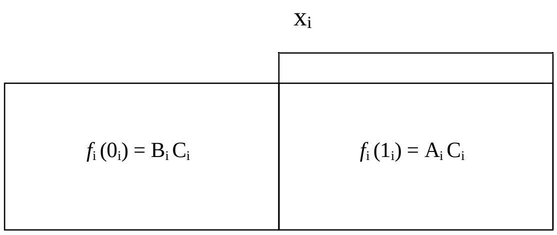


Fig. (10). A VEKM representation of the function $f_i = f_i\,(X_1, X_2, ..., X_i)$.

Now employing the incompletely-specified definitions (7c) and (7d) for s_i and t_i, we obtain

$$s_i \quad {}_i \; {}_i \; B_i \;\; d(\, B_i \;\; C_i\,) \tag{24}$$

$$t_i \quad {}_i \quad {}_i \;\; d(\, B_i\, C_i\,) \tag{25}$$

Figure (11) displays conventional Karnaugh maps for s_i and t_i as functions of the coefficients A_i, B_i, and C_i. It indicates that we can simplify the expressions of s_i and t_i to

$$s_i = B_i, \tag{26}$$

$$t_i \quad {}_i, \tag{27}$$

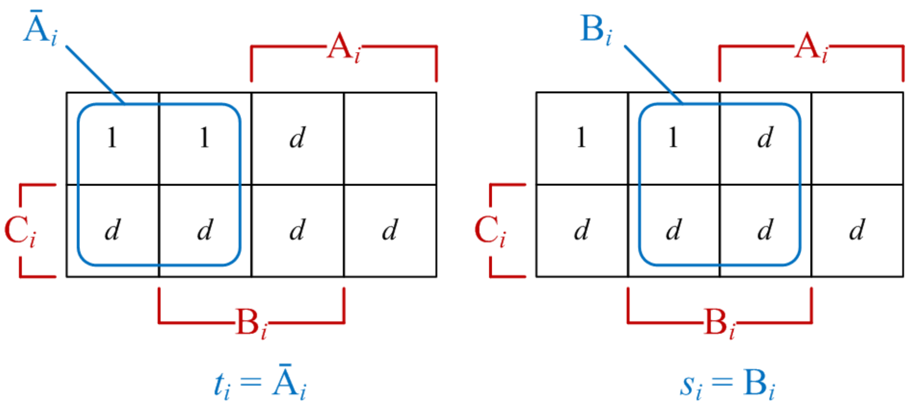


Fig. (11). Conventional Karnaugh maps of s_i and t_i as functions of A_i, B_i and C_i.

and hence, we can replace (5b) by:

$$B_i \;\leq\; {}_i \;\leq\; {}_i, \qquad i = n, (n\text{-}1), ..., 2, 1 \tag{28}$$

In agreement with formula (13) of Rudeanu Second Method. We note that the freedom allowed by the conditions in (7) is not fully utilized in Rudeanu Second Method in general. In fact, this method strives to achieve local minimality over all possible choices of the coefficients A_i, B_i and C_i. This kind of minimality is sub-optimal compared with global minimality over the underlying set of variables X_1, X_2, ..., X_i and algebra generators a and b, that is suggested by (7) and fully employed in our subsumptive VEKM solution (Section3). Though the Rudeanu Second Method is suboptimal, it achieved the minimal solution for the example in [1, 20] and for Example 1 herein. However, it failed to obtain minimality for our Example 2.

6. Parametric General Solutions

Brown [1] proved that n parameters are sufficient to construct a parametric general solution of an n-variable Boolean equation $g(\mathbf{X}) = 1$, where g: $\mathbf{B}^n \to \mathbf{B}$. He proposed a procedure for constructing such a solution using the fewest possible parameters, p_1, p_2, …, p_k, which are elements of **B**, where $k \leq n$. In [25, 26], we adapted this procedure of Brown into a VEKM procedure as follows:

(a) Construct a VEKM representing $g(\mathbf{X})$. Such a construction is achieved via a Boole-Shannon tree expansion [1]. If the original Boolean equation is in the dual form $f(\mathbf{X}) = 0$, then construct a VEKM for $f(\mathbf{X})$, and complement it cell-wise [27] to obtain a VEKM for $\bar{f}(X) = g(X)$.

(b) Expand the entries of the VEKM of $g(\mathbf{X})$ as ORing of appropriate atoms of the Boolean carrier **B,** or equivalently as a minterm expansion of the free Boolean algebra of **B.**

(c) If certain atoms of **B** do not appear at all in any cell of the VEKM for $g(\mathbf{X})$, then these atoms must be forbidden or nullified. Such nullification constitutes *a consistency condition* for the given Boolean equation.

(d) Construct a VEKM for an associated function $G(X_1, X_2, \ldots, X_n; p_1, p_2, \ldots, p_k)$. This VEKM is deduced from that of $g(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n)$ through the following modifications:

(d1) Each appearance of an entered atom in the VEKM of g is ANDed with a certain element of a set of orthonormal tags of minimal size. An orthonormal set consists of a set of terms T_i, i = 1, 2, …, k, which are both exhaustive ($T_1 \vee T_2 \vee \ldots \vee T_k = 1$) and mutually exclusive ($T_i T_j = 0$ for $1 \leq i < j \leq k$).

(d2) Each nullified atom is entered as a don’t care in all the VEKM cells.

(e) The parametric solution is

X_i = The sum (ORing) of the 2^{n-1} cells constituting half of the VEKM in which $\mathbf{X}_i$ is asserted ($\mathbf{X}_i = 1$), i = 1, 2, …, n. (29)

(f) (f) Apply a VEKM minimization procedure [28-31] to recast (29) in a minimal form.

Example 1(revisited):

We apply the aforementioned technique to the function g in Fig. (12). Which is the complement of *f* given by Eq. (8) or Figure (1).Steps of the solution are illustrated by Figs. (13 and 14), where the set of orthonormal tags (p_1 $_2p_3$, p_1 $_2$ $_3$, $p_1p_2p_3$, p_1p_2 $_3$, $_1p_2p_3$, $_1p_2$ $_3$, $_1$ $_2$) is used for atom a and the set of orthonormal tags (p_2 $_3$ $_2$, p_2p_3) is used for atom . The final parametric solution is simplified via the VEKMs in Fig. (15) and are given by

$$X_1 \quad _1 \; _2 \quad _1p_3 \; ap_1p_2 \; _3 \quad _2 \; _3$$

$$X_2 \; _1 \; _2 \; _1 \; _3 \; _2 \; _3 \; _2 \; _3$$

$$X_3 \quad _1 \; p_2p_3 \; , \tag{30}$$

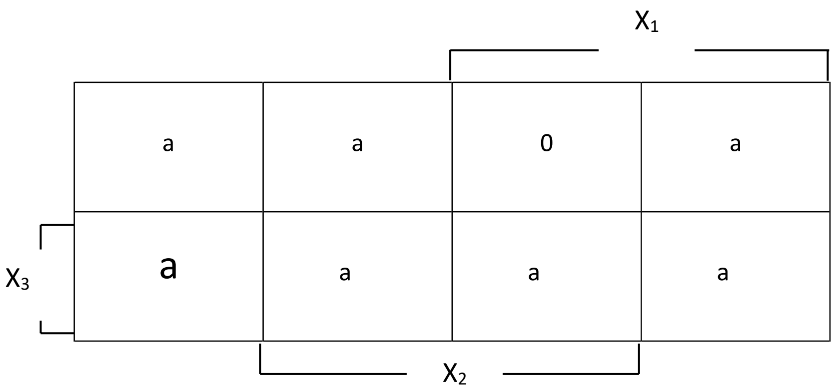
Together with the consistency condition

$$0 = 0 \tag{31}$$

			X_1	X_1
	a	a	0	a
X_3	a	1	1	1
		X_2	X_2	

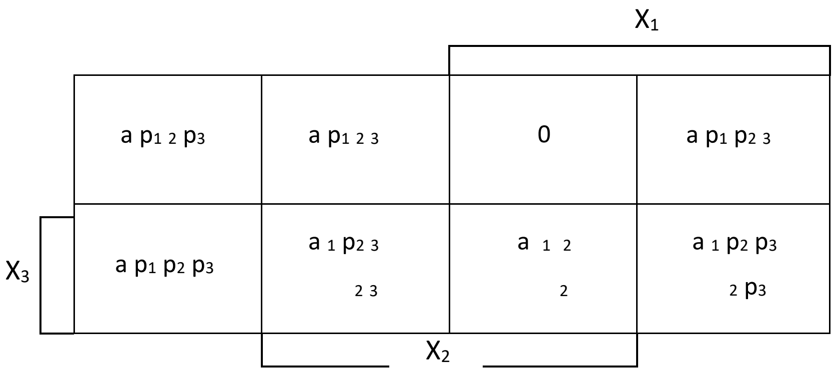
g =

Fig. (12). A natural map representation of the Boolean function g, the complement of the function *f* in Fig.(1).



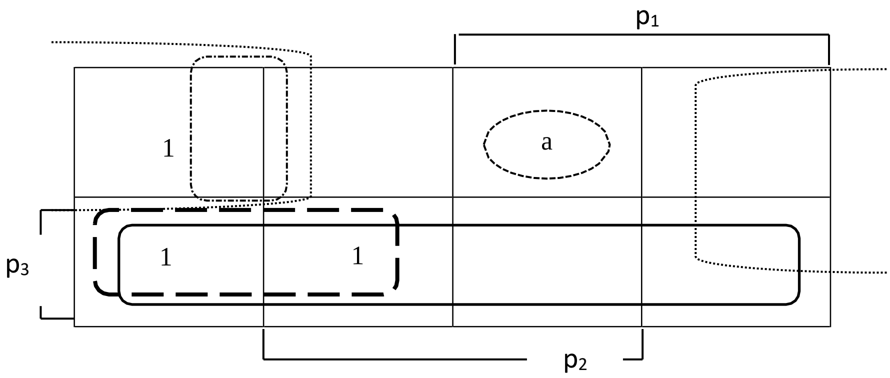
g (X₁, X₂, X₃,)

Fig. (13). Entries of the map for the function g in Fig.(12) expanded in terms of atoms of B_4 or minterms of *FB(a)*.

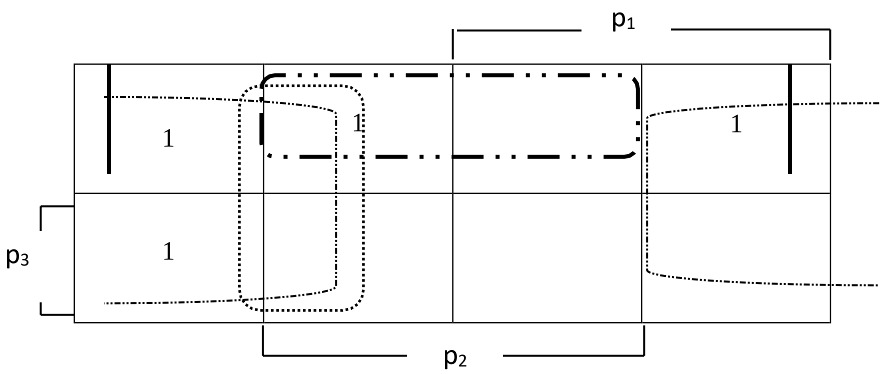


$G\ (X_1, X_2, X_3;\ \mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3)$

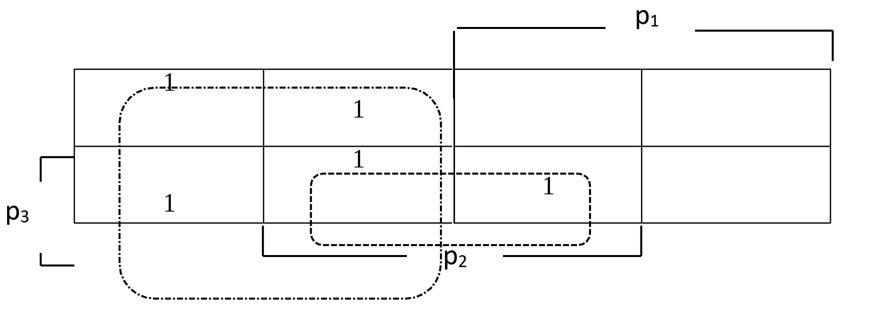
Fig. (14). Each appearance of an entered atom in Fig.(13) is ANDed with a certain element of a set of orthonormal tags.



$X_1 \quad {}_1 \quad {}_2 \quad {}_1p_3\ ap_1p_2 \quad {}_3 \quad {}_2\ {}_3$



$X_2 \quad {}_1 \quad {}_2 \quad {}_1 \quad {}_3 \quad {}_2 \quad {}_3 \quad {}_2\ {}_3.$



$X_3 \quad {}_1\ p_2p_3$

Fig. (15). VEKM expression of the parametric solution.

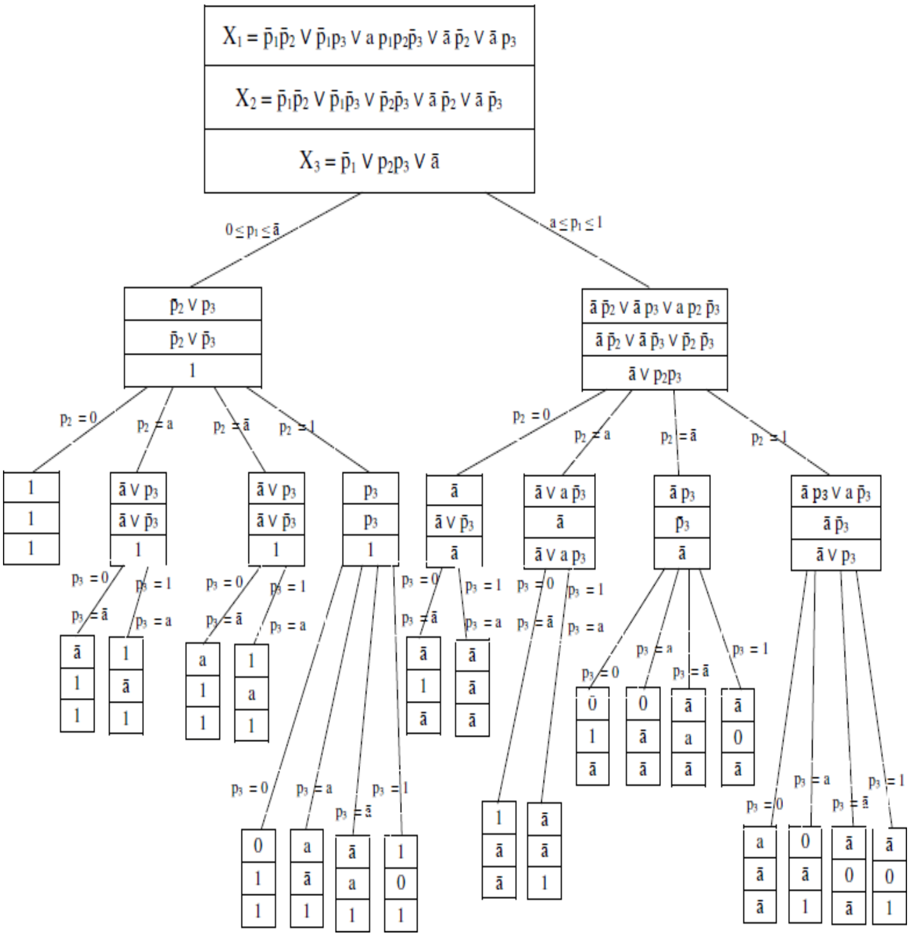


Fig. (16). Tree used to deduce all 21 particular solution of (9) from the parametric solutionn (30).

Figure (16) shows the tree used to deduce all 21 particular solutions from the parametric solutions (30). These are the same solutions as those in Fig.(3). It can be seen from Figure (14) that the particular solution corresponding to tagging atom a with tag p_1 $_2p_3$ and tagging atom with tag p_2 $_3$ is $X_1 = 0$, $X_2 = X_3$. This solution must satisfy

$$p_1 \quad _2p_3\, a = a, \tag{32a}$$

$$p_2 \quad {}_3 \qquad , \tag{32b}$$

or equivalently

$$a \ \leq \ p_1 \quad {}_2 p_3 \tag{33a}$$

$$\leq \ p_2 \quad {}_3 \tag{33b}$$

Figure (16) demonstrates that this solution is the parametric values indeed obtained for the parameters {$a \leq p_1 \leq 1$, p_2 , $_3 = a$}, which satisfy (32) and (33).

Example 2 (revisited):

We apply the VEKM procedure to the function g in Fig. (17) which is the complement of *f* given by Eq. (10) or Figure (4). The main step of the solution is illustrated by Fig. (18), where an orthonormal set { , } is used to tag appearances of each of the three asserted atoms b , b, and ab .The parametric solution is given by

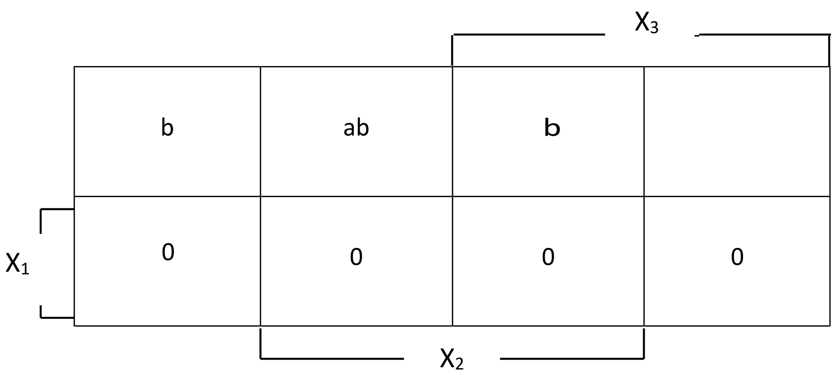
$$X_3 \quad b \qquad ,$$

$$X_2 \ b \ a \ ,$$

$$X_1 = 0, \tag{34}$$

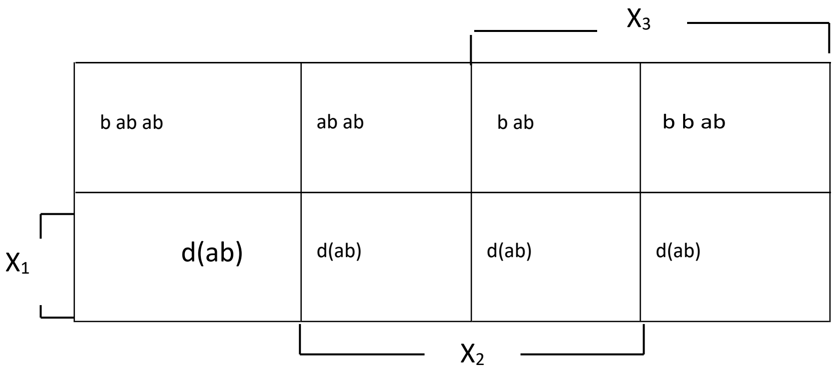
together with the consistency condition

$$ab = 0. \tag{35}$$



g =

Fig. (17). A natural map representation of the Boolean function g, the complement of the function *f* in Fig.(3).



$G\,(X_1, X_2, X_3;\ p_1, p_2, p_3)$

Fig. (18). Each appearance of an entered atom in Fig.(17) is ANDed with a certain element of the set of orthonormal tags while the atom ab **that appears nowhere in Fig. (17) is entered don't care.**

Figure (19) shows the tree used to deduce all 8 particular solutions of g = 1 from the parametric solutions (34) subject to (35). These are the same solutions as those in Fig.(6). It is clear from Fig. (18) that the particular solution corresponding to tagging atom b with , atom ab with p, and atom b with is $X_1 = 0$, $X_2 = a\ \ b$, $X_3 = b$. This solution must satisfy

$$b \quad b\,, \tag{36a}$$

$$ab \quad ab\,, \tag{36b}$$

$$b \quad b, \tag{36c}$$

or, equivalently

$$b \ \leq \quad , \tag{37a}$$

$$ab \ \leq\ p, \tag{37b}$$

$$b \ \leq \quad , \tag{37c}$$

Figure (19) demonstrates that this solution is indeed obtained for the parametric value p = a which satisfies (36) or (37) together with (35).

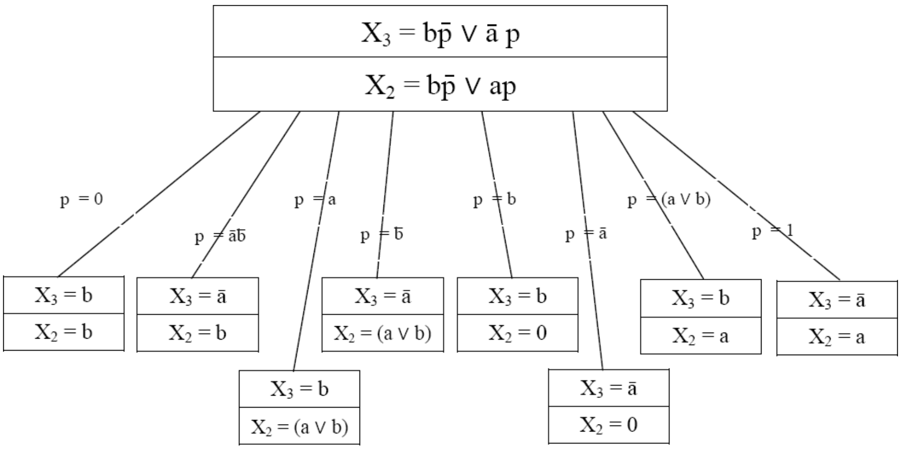


Fig. (19). Tree used to deduce all particular solutions from the general parametric solutions (34) subject to (35). The eight values assigned to p are from the collapsed lattice in Fig.(8). (X_1 = 0, ab = 0).

7. Conclusions

In this paper, we presented a tutorial exposition and comparison of the main types of Boolean-equation solutions, namely, subsumptive general solutions, parametric general solutions, and particular solutions. We also made a detailed comparison of the two prominent classes of methods used in the solution of Boolean equations, viz. , the class of purely-algebraic methods and the class of tabular or map methods. Though map methods can be generally categorized as tabular methods, they have their own distinguishing characteristics to warrant classifying them as a separate class of methods.

We used techniques employing the Variable-Entered Karnaugh Map (VEKM) as representatives of map methods, since the VEKM is the natural map for the underlying finite Boolean algebras. We pointed out that these techniques are semi-algebraic in nature and include purely-algebraic techniques as special limiting cases. Hence, we anticipated that VEKM techniques should never be inferior to algebraic techniques. Later, we demonstrated that VEKM techniques are occasionally superior to algebraic techniques since the former techniques naturally and easily secure minimality over the basic set of pertinent variables and generators, while the latter techniques seek a restricted sort of minimality over a set of chosen coefficients. We supported our argument by an illustrative example in which a VEKM technique was *easier to implement* and produced a *more compact solution* (minimal solution) that was much *easier to expand* as a tree of particular solutions. This example was over $\mathbf{B}_{16} = FB(a, b)$ which collapsed to $\mathbf{B}_8$. Another supporting example that is given in [32] is over $\mathbf{B}_{65536} = FB(a, b, c, d)$ that collapsed to $\mathbf{B}_{32}$.

In conclusion, we note that algebraic methods are useful in the initial study of the subject to maintain the rigor and set the theoretical framework. Map methods (VEKM methods, in particular) complement algebraic methods, as they provide pictorial insights, require easier shortcut manipulations, and produce much more compact general solutions.

8. Acknowledgment

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مقارنة بين الطرائق الجبرية والطرائق الخريطية المستخدمة في حل المعادلات البولانية

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ملخص البحث. يمكن تصنيف طرائق حل المعادلات البولانية على الإجمال بأنها طرائق جبرية أو جدولية أو عددية أو خريطية. إلا أن الأبرز بين أصناف الطرائق هذه هو صنفا الطرائق الجبرية والخريطية. إن ورقة البحث هذه تستعرض هذين الصنفين الأبرز من تلك الطرائق و تقارنهما من ناحية درجة البساطة أو التعقيد ودرجة السرعة أو الكفاية ومدى القابلية للاستخدام. تحدد الورقة الأنواع الرئيسة لحلول المعادلات البولانية بأنها (أ) الحلول العامة الاحتوائية، و(ب) الحلول العامة المعلمية، و(ج) الحلول الخاصة. في الحل العام الاحتوائي يتم حصر كل من المتغيرات في فترة من القيم اعتمادا على توليد تتابع من الحاذفات العطفية أو الاتحادية للدالة الأصلية. أما في الحل العام المعلمي فيتم التعبير عن كل المتغيرات بدلالة معامل اختيارية وهي قيم يتم اختيارها من عناصر الجبر البولاني الحاكم للمسألة، بينما يمثل الحل الخاص تعيينا لكل متغير بقيمة محددة تؤخذ من عناصر الجبر البولاني الحاكم بحيث تؤدي هذه التعيينات في مجموعها إلى تحويل المعادلة البولانية إلى متطابقة. توفر الورقة شرحا تعليميا لصنفي طرائق الحل الأساسيين ولكل نوع من أنواع الحلول المذكورة بدلالة مثالين توضيحيين صغيرين. يظهر جليا أن الطرائق الخريطية تنافس الطرائق الجبرية (وتتفوق عليها أحيانا) لأنها تتمتع بسيطرة أفضل على التصغير الأعظمي لتمثيل الدوال المعنية، ومن ثم فإنها أقدر على إنتاج حلول عامة معلمية واحتوائية أصغر وأبسط.